Fuzzy Model Reference Adaptive Control System

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Abstract:– When using the Lyapunov synthesis approach to construct a stable fuzzy control system, one important way is to regard the fuzzy systems as approximators to approximate the unknown functions in the system to be controlled. Concerning the unknownness of the unknown functions, generally there are two cases: a completely unknown case, and a partly unknown case. However, most of the schemes presented so far have only focused on the former. Clearly, if an unknown function belongs to the latter, the knowledge available about the function should be utilized as much as possible in the development of the control system. In this paper, our goal is to design a fuzzy controller for a class of model reference adaptive systems with uncertainties, which can correspond to the either case. Also, we propose a unique way to deal with the uncertainties, i.e., adopt a switching function with an alterable coefficient, which is tuned by adaptive law based on the tracking error.

Key-Words:– Lyapunov function, fuzzy approximator, MRACS, stability, adaptive law.

1 Introduction

The objective of a controller is to drive the output of a plant to keep at reference value or to follow another signal. To this end and under some circumstances where system knowledge and dynamics models in the traditional sense are uncertain and time varying, fuzzy control has appeared strongly capable in a large number of research and industrial applications. However, some researchers who stick on the traditional control system where the proven system stability is the first task to be considered casted a skeptical look sometimes on the system due to the lack of formal synthesis technic that can guarantee system stability among other basic requirements for a control system. This is the major reason why a large number of research has focused on fuzzy control systems with a keyword of "system stability" since earlier 1990's (e.g. [1]-[10] and references therein). In such a fuzzy control system, the Lyapunov synthesis approach is used to construct a stable controller, and to deal with the uncertainties in the system, the traditional adaptive control theory is merged with fuzzy approximation theory [2] where the unknown functions in the system are approximated by parameterized fuzzy approximators.

In general, there are two kinds of uncertainties in a system to be controlled. One is caused by the lack of the system structure and information, and another one is the so-called disturbances. In order to deal with the one of uncertainties for lack about the system, the approaches developed so far are almost same, *i.e.,* using the fuzzy approximators to approximate the corresponding functions. Obviously, before using a fuzzy approximator to approximate an unknown function, the extent of the unknownness should be examined. Generally, there are two cases: a completely unknown case, and a partly unknown case. Actually, most of the schemes presented so far have only focused on the former, and few studies pay attention to the latter. In a system to be controlled, if an unknown function belongs to the latter, the knowledge available about the function, clearly, should be utilized to the maximum in the development of the control system. Although some papers [4],[5] focused such a problem, the proposed schemes did not involve the control gain, which is not a trivial problem indeed in a control system. On the other hand, to deal with the one of uncertainties with disturbances which may come from either the inside or the outside system, the upper bounds of uncertainties are assumed to be known as well as the reconstruction errors between the optimal approximators and their corresponding functions to be approximated [9]. In fact, such a kind of the upper bounds about the uncertainties is not easy to be known in advance of designing the control system, therefore, to be *safe* for the system stability, a larger magnitude of assumption about the uncertainties is opposed, and finally, a bigger chattering in control inputs is always concerned. Here in this paper we employ the idea that such an upper bound is tuned by adaptive law [5].

In this paper, our goal is to design a fuzzy controller for a class of model reference adaptive systems with uncertainties in which any type of the uncertainties can be corresponded. Also, to deal with the uncertainties, we adopt a switching function with an alterable coefficient, which is tuned by adaptive law based on the tracking error, in stead of using the upper bound assumptions. The adaptive law to adjust all parameters will be developed based on the Lyapunov synthesis approach. It is shown that the proposed fuzzy controller guarantees tracking error, between the states of the considered system and its reference model, to be uniformly bounded, also the bound can be made arbitrarily small by choosing appropriately related parameters, while maintaining all signals in the system asymptotically stable. Finally, the control performance will be confirmed by a computer simulation.

2 Problem Statement

This paper focuses on the design of adaptive fuzzy control algorithms for a class of nonlinear systems whose equation of motion can be expressed in the canonical form:

$$\dot{x}_1(t) = x_2$$
$$\cdots$$
$$\dot{x}_{n-1}(t) = x_n$$
$$\dot{x}_n(t) = \mathbf{a}_1(X)^T X + b_1(X)u(t) + d(t) \qquad (1)$$

where $X^T = [x_1\ x_2\ \cdots\ x_n] = \left[x\ \dot{x}\ \cdots\ x^{(n-1)}\right]$ is the system state; $u(t)$ is the control input; $\mathbf{a}_1(X) \in R^{n\times 1}$, and $b_1(X)$ are smooth vector function, scalar function, respectively; $d(t)$ denotes the disturbance in the system. In the above system, generally functions $\mathbf{a}_1(X), b_1(X)$ are not known as well as $d(t)$. However, there is a case that they can be partly known prior to developing the control system. In this way, the knowledge about the functions, clearly, should be utilized as much as possible in the development of the control system to improve the control performance. Therefore, system (1) can be rewritten as,

$$\dot{X}(t) = (A_0 + A)\,X + B(b_0 + b(X))u + Bd(t) \qquad (2)$$

where,

$$A_0 = \begin{bmatrix} 0 & I_{n-1} \\ & \mathbf{a}_0^T & \end{bmatrix} \in R^{n\times n},$$

$$A = \begin{bmatrix} 0 & 0_{n-1} \\ & \mathbf{a}^T(X) & \end{bmatrix} \in R^{n\times n}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} \in R^{n\times 1}$$

$\mathbf{a}_0 \in R^{n\times 1}$, and b_0 are the known parts in $\mathbf{a}_1(X)$, and $b_1(X)$, respectively, which will be used in the controller structure directly, and $\mathbf{a}(X) \in R^{n\times 1}$, and $b(X)$ are the unknown parts in $\mathbf{a}_1(X)$, and $b_1(X)$, respectively. If $\mathbf{a}_1(X)$ or $b_1(X)$ in (1) is completely unknown previously, $\mathbf{a}_0$ or b_0 simply becomes 0. Clearly, form (2) can correspond to either case: $\mathbf{a}_1(X)$ or $b_1(X)$ is completely unknown or partly unknown.

A stable, controllable reference model is given as follows,

$$\dot{X}_m(t) = A_m X_m + B_m r \qquad (3)$$

where $X_m \in R^{n\times 1}$ is the state vector of the reference model; and A_m, B_m are given by

$$A_m = \begin{bmatrix} 0 & I_{n-1} \\ & \mathbf{a}_m^T & \end{bmatrix} \in R^{n\times n},$$

$\mathbf{a}_m^T = [a_{m1}\ \cdots\ a_{mn}] \in R^{1\times n}$, $B_m^T = [0\ \cdots\ 0\ b_m] \in R^{1\times n}$.

The problem we consider in this paper is to design a controller $u(t)$ for (2) which ensures the system state $X(t)$ follows the reference model state $X_m(t)$, namely, tracking error vector $E(t) \in R^{n\times 1}$, which is defined by

$$E(t) = X(t) - X_m(t) \qquad (4)$$

is uniformly bounded, also the boundary can be made arbitrarily small by choosing appropriately related parameters, while maintaining all signals in the system asymptotically stable.

The nonlinear functions $\mathbf{a}$ and b in (2) are unknown, so before developing our control algorithm we have to solve the problem of approximating $\mathbf{a}(X)$ and $b(X)$. In the following section, it will be shown that using fuzzy IF-THEN rules, the unknown functions $\mathbf{a}$ and b can be approximated by some parameterized fuzzy approximators.

To proceed with our development, we state our assumptions on the system.

Assumption : The control gain is finite, nonzero, and of known sign for all X; without loss of generality this sign can be taken as positive, *i.e.,* $b_0 \geq 0,\ b(X) > 0$. In addition, the uncertainty $d(t)$ is bounded.

Remark 1 As the regular assumptions postulated on the control gain b [9],[13], *a priori* upper bound of its variation, $|\frac{d}{dt}b(X)|$, should be known. Here in this paper we remove it from the assumption.

Remark 2 Compared with other schemes, there is an important difference about the assumption postulated on the uncertainty. In this paper, we just suppose that the uncertainty is bounded, and the value of real boundary does not need to be known.

3 Adaptive Fuzzy Controller

3.1 Fuzzy Approximator

Fuzzy model addresses the imprecision of the input and output variables directly by defining them with fuzzy sets in the form of membership functions. The basic configuration of fuzzy model includes a fuzzy rules base, which consists of a collection of IF-THEN fuzzy rules. Now, we consider a fuzzy model with *singleton consequent, product inference*, *Gaussian membership function* in the antecedent, and *central average defuzzifier*, hence, such a fuzzy model can be written as

$$\mathcal{F}(Z) \quad = \quad W^T \cdot G(Z) \qquad (5)$$

where $Z^T = [z_1, z_2, \ldots, z_n]$, $W^T = [w_1, w_2, \ldots, w_N]$ with N being the number of fuzzy rules; $G^T(Z) = [g_1(Z), g_2(Z), \ldots, g_N(Z)]$ with $g_j(Z) = \frac{\prod_{i=1}^n \mu_{A_j^i}(z_i)}{\sum_{j=1}^N \prod_{i=1}^n \mu_{A_j^i}(z_i)}$ where $\mu_{A_j^i}(z_i)$ is a Gaussian membership function, defined by

$$\mu_{A_j^i}(z_i) = \exp\left[-\left(\frac{z_i - \xi_j^i}{\sigma_j^i}\right)^2\right] \qquad (6)$$

where ξ_j^i indicates the position, and σ_j^i indicates the variance of the membership function.

We now can show an important property of the fuzzy system above. As shown by Wang *et al* [8], the fuzzy system has the same pattern as a neural network. Exactly as a neural network, which has powerful abilities of learning and approximation, a fuzzy system with the Gaussian membership is capable of uniformly approximating any well-defined nonlinear function over a compact set U to any degree of accuracy. The following theorem theoretically supports this claim.

Theorem 1 [15] *For any given real continuous function f on the compact set $U \in \mathcal{R}^n$ and arbitrary ε^*, there exists an optimal fuzzy system expansion $\mathcal{F}^*(Z) = W^{*T} \cdot G(Z)$ such that*

$$\sup_{Z \in U} |f - \mathcal{F}^*(Z)| \leq \varepsilon^* \tag{7}$$

This theorem states that the fuzzy system (5) is a universal approximator on a compact set. In this paper, we use the terms *fuzzy universal approximator* or *fuzzy approximator* to refer to the fuzzy system. Since the fuzzy universal approximator is characterized by parameter vectors $W(t)$, the optimal $\mathcal{F}^*$ does contain an optimal vector W^*. Notice that even though fuzzy approximator is linear with respect to the adjustable parameter vector $W(t)$, we may, e.g., approximate a function $f(Z) = a + \cos(bZ^T Z)$ which is not linear in an independent set of parameters $[a, b]$ [6]. Thus we are using such a fuzzy approximator which is linear in the parameters to describe functions which are not necessarily linear in another set of parameters.

3.2 Design of Controller

Differentiating two sides of tracking error (4) and combining (2), (3) with it yields

$$\begin{aligned} \dot{E}(t) &= \dot{X} - \dot{X}_m \\ &= A_m E - B\left[(\mathbf{a_m} - \mathbf{a_0})^T X + b_m r \right. \\ &\quad \left. - \mathbf{a}(X)^T X - (b_0 + b(X))u - d\right] \end{aligned} \tag{8}$$

On the other hand, since A_m in (3) is stable, *i.e.,* all the eigenvalues A_m have negative real parts, for every positive definite matrix $Q \in R^{n\times n}$, the following Lyapunov matrix equation

$$A_m^T P + P A_m = -Q \tag{9}$$

has a unique solution $P \in R^{n\times n}$ that is also positive definite [12]. Therefore, it is not difficult to design a controller like $u = (b_0 + b(X))^{-1}\left[(\mathbf{a_m} - \mathbf{a_0} - \mathbf{a}(X))^T X + b_m r - d\right]$ which ensures that $\lim_{t\to\infty} E(t) \to 0$ if $\mathbf{a}(X), b$, and d in (8) are available, because for Lyapunov function $v(t) = \frac{1}{2}E^T P E$, it is easy to show that $\dot{v}(t) \leq -\frac{1}{2}E^T Q E$. However, as mentioned before, the functions $\mathbf{a}$, and b are supposed to be unknown as well as disturbance d in this paper. Therefore, the problem is how to determine a controller $u(t)$ when the system involves such a kind of uncertainties. Thus we have to approximate them to proceed with the development. Here the fuzzy approximator described in the previous subsection is employed. Let $\mathbf{a}^*(X) = \mathbf{W}_a^{*T}\mathbf{G}_a(X), b^*(X) = W_b^{*T} G_b(X)$ be the *optimal fuzzy approximators* of the unknown functions $\mathbf{a}(X), b(X)$, respectively. According to Theorem 1, there are a small positive vector $\Phi_a^* \in R^{n\times 1}$, and a small positive value ϕ_b^* such that, the errors,

$$\begin{aligned} \Phi_a &= \mathbf{a}(X) - \mathbf{a}^*(X) &(10)\\ \phi_b &= b(X) - b^*(X) &(11) \end{aligned}$$

which are referred to as *reconstruction errors*, satisfy the following inequalities,

$$\begin{aligned} |\Phi_a| &\leq \Phi_a^* &(12)\\ |\phi_b| &\leq \phi_b^* &(13) \end{aligned}$$

Given that b^* is the optimal fuzzy approximator of b, it is reasonable to consider that,

$$b^* \leq \bar{b} \tag{14}$$

where the upper bound $\bar{b}$ is avaliable. We also should note that the bounds of Φ_a^*, ϕ_b^* do not need to be known in this paper. Apparently, the *optimal parameters*, $\mathbf{W}_a^*$ and W_b^* in the optimal fuzzy approximators are unknown either, therefore, as usual, their estimates, denoted as $\hat{\mathbf{a}}(X) = \hat{\mathbf{W}}_a^T(t)\mathbf{G}_a(X), \hat{b}(X) = \hat{W}_b^T(t) G_b(X)$, are adopted, and which will be tuned based on the tracking error E.

Let's define a new function,

$$s_\phi = E^T P B - \phi \cdot \mathrm{sat}\left(\frac{E^T P B}{\phi}\right) \tag{15}$$

where, ϕ is a little constant which describes the width of a boundary layer and is used to prevent discontinuous control transitions, and sat$(\cdot)$ is saturation function defined by

$$\mathrm{sat}(x) = \begin{cases} 1, & \text{if } \ x > 1 \\ x, & \text{if } \ |x| \leq 1 \\ -1, & \text{if } \ x < -1 \end{cases} \tag{16}$$

With the above in mind, now we are ready to develop our control system. Referring to the controller when $\mathbf{a}(X), b(X)$, and d are known, our adaptive fuzzy controller is determined by,

$$\begin{aligned} u = {} & (b_0 + \hat{b}(X))^{-1}\Big[b_m r + (\mathbf{a_m} - \mathbf{a_0} - \hat{\mathbf{a}}(X))^T X \\ & - (\hat{\Phi}_a^T |X| + (\hat{\phi}_b + \delta_s)|u_0| + \hat{d})\mathrm{sat}(s_\phi)\Big] \end{aligned} \tag{17}$$

with

$$\delta_s = \begin{cases} 0, & \text{if } \hat{W}_b \in \mathcal{S} \\ |\hat{b}(X)| + \bar{b}, & \text{otherwise} \end{cases} \tag{18}$$

$$\mathcal{S} = \{\hat{W}_b \in \mathcal{R}^m \mid \hat{W}_b^T G_b(X) > 0\} \tag{19}$$

where $\hat{\Phi}_a \in R^{n\times 1}$, and $\hat{\phi}_b$ are the estimates of Φ_a^*, and ϕ_b^*, respectively; $\hat{d}$ is the estimate of the uncertainty boundary, in symbols d^*; u_0 is the one-step-previous input. The role of adopting $\hat{\Phi}_a, \hat{\phi}_b$ is not only to avoid *a prior* knowledge about the reconstruction errors, but also to make compensation for their approximation errors. $\hat{\Phi}_a, \hat{\phi}_b$, and $\hat{d}$ are estimated by,

$$\dot{\hat{\Phi}}_a = -\sigma(\hat{\Phi}_a - \Phi_{a0}) + |s_\phi X| \tag{20}$$

$$\dot{\hat{\phi}}_b = -\sigma(\hat{\phi}_b - \phi_{b0}) + |s_\phi u| \quad (21)$$

$$\dot{\hat{d}} = -\sigma\hat{d} + |s_\phi| \quad (22)$$

where $\sigma > 0$ is a leakage constant [16], which counteracts a drift of parameter values into regions of instability in the absence of persistent excitation; Φ_{a0}, ϕ_{a0} are *a priori* estimates (or information) of $\hat{\Phi}_a, \hat{\phi}_a$, respectively. Of course, such estimates, which's roles will be made clear later, shrink to zero if they are not unavailable.

The adaptive law is synthesized by,

$$\dot{\hat{\mathbf{W}}}_a = -\sigma\left(\hat{\mathbf{W}}_a - \mathbf{W}_{a0}\right) + s_\phi X \mathbf{G}_a^T(X) \quad (23)$$

$$\dot{\hat{W}}_b = \begin{cases} -\sigma\left(\hat{W}_b - W_{b0}\right) + s_\phi u G_b(X), & \text{if } \hat{W}_b \in \mathcal{S} \\ 0, & \text{otherwise} \end{cases} \quad (24)$$

where $\hat{\mathbf{W}}_a$, and $\hat{W}_b$ are the estimates of $\mathbf{W}_a^*$, and W_b^*, respectively; correspondingly, $\mathbf{W}_{a0}$, and W_{b0} are *a priori* estimates of $\hat{\mathbf{W}}_a$, and $\hat{W}_b$, respectively.

Remark 3 As a matter of fact, it does not need the switching function(18), in the sense of simplifying the approach by using the information of $\bar{b}$ in (14) in both inside and outside of $\mathcal{S}$. However, such an information, $\bar{b}$, is often not being known well, so it is usually set large enough for the reason of stability. Here in this paper, the information is only used in outside $\mathcal{S}$. It means we do not want to use it unless the system stability is threatened. In addition, the switching adaptive law (24) serves for both purposes of avoiding the risk of a zero-denominator calculation in (17), and system stability that will be shown later.

3.3 Analysis of System Stability

We begin the stability analysis by defining a Lyapunov function as

$$v = \frac{1}{2}\left(E^T P E + \sum_{i=1}^{n}\sum_{j=1}^{N}\tilde{w}_{a_{ij}}^2 + \tilde{W}_b^T\tilde{W}_b + \tilde{\Phi}_a^T\tilde{\Phi}_a + \tilde{\phi}_b^2 + \tilde{d}^2\right) \quad (25)$$

where $\tilde{\mathbf{W}}_a = \left[\tilde{w}_{a_{ij}}\right]_{n\times N}$, and

$$\tilde{w}_{a_{ij}} = \hat{w}_{a_{ij}} - w_{a_{ij}}^* \quad (26)$$

$$\tilde{W}_b = \hat{W}_b - W_b^* \quad (27)$$

$$\tilde{\Phi}_a = \hat{\Phi}_a - \Phi_a^* \quad (28)$$

$$\tilde{\phi}_b = \hat{\phi}_b - \phi_b^* \quad (29)$$

$$\tilde{d} = \hat{d} - d^* \quad (30)$$

Differentiating the first right-hand term, and substituting (8)-(11), (28), and (29) into which yields,

$$\begin{aligned} & E^T P \dot{E} \\ = & -\frac{1}{2}E^T Q E - E^T P B\Big[(\mathbf{a_m} - \mathbf{a_0} - \hat{\mathbf{a}}(X))^T X + b_m r \\ & -\Phi_a^T X + \tilde{\mathbf{a}}^T X - (b_0 + \hat{b}(X))u - \phi_b u + \tilde{b}u - d\Big] \end{aligned} \quad (31)$$

where $\tilde{\mathbf{a}} = \hat{\mathbf{a}}(X) - \mathbf{a}^*(X)$, $\tilde{b} = \hat{b}(X) - b^*(X)$. As shown in (15), since ϕ is a constant with small value, a gap between s_ϕ and $E^T PB$ is within a small magnitude. Also, the gap can be covered by disturbance d. In other word, the gap can be viewd as a part of the disturbance, which is unknown but dealt with by an adaptive law in this paper. Under this consideration, (31) can be rewritten as

$$\begin{aligned} & E^T P \dot{E} \\ = & -\frac{1}{2}E^T Q E - s_\phi\Big[(\mathbf{a_m} - \mathbf{a_0} - \hat{\mathbf{a}}(X))^T X + b_m r \\ & -\Phi_a^T X + \tilde{\mathbf{a}}^T X - (b_0 + \hat{b}(X))u - \phi_b u + \tilde{b}u - d\Big] \end{aligned} \quad (32)$$

Taking the time derivative of the Lyapunov function of (25), and substituting (31) into it, we have,

$$\begin{aligned} \dot{v} \le & -\frac{1}{2}E^T Q E - s_\phi\Big[(\mathbf{a_m} - \mathbf{a_0} - \hat{\mathbf{a}}(X))^T X \\ & + b_m r + \tilde{\mathbf{a}}^T X - (b_0 + \hat{b}(X))u + \tilde{b}u\Big] \\ & + |s_\phi|\Phi_a^{*T}|X| + |s_\phi|\phi_b^*|u| + |s_\phi|d^* \\ & + \sum_{i=1}^{n}\sum_{j=1}^{m}\tilde{w}_{aij}\dot{\hat{w}}_{aij} + \tilde{W}_b^T\dot{\hat{W}}_b \\ & + \tilde{\Phi}_a^T\dot{\hat{\Phi}}_a + \tilde{\phi}_b\dot{\hat{\phi}}_b + \tilde{d}\dot{\hat{d}} \end{aligned} \quad (33)$$

Substituting (17), (20)-(23) into (33), it follows that

$$\begin{aligned} \dot{v} \le & -\frac{1}{2}E^T Q E - s_\phi\tilde{b}u \\ & -(\hat{\Phi}_a^T|X| + (\hat{\phi}_b + \delta_s)|u_0| + \hat{d})|s_\phi| \\ & + |s_\phi|\Phi_a^{*T}|X| + |s_\phi|\phi_b^*|u| + |s_\phi|d^* \\ & -\sigma\sum_{i=1}^{n}\sum_{j=1}^{m}\tilde{w}_{aij}(\hat{w}_{aij} - w_{aij0}) + \tilde{W}_b^T\dot{\hat{W}}_b \\ & -\sigma\tilde{\Phi}_a^T(\hat{\Phi}_a - \Phi_{a0}) + \tilde{\Phi}_a^T|s_\phi X| \\ & -\sigma\tilde{\phi}_b(\hat{\phi}_b - \Phi_{b0}) + \tilde{\phi}_b|s_\phi u| \\ & -\sigma\tilde{d}\hat{d} + \tilde{d}|s_\phi| \end{aligned} \quad (34)$$

where the fact that $s_\phi \cdot \mathrm{sat}(s_\phi) = |s_\phi|$ is used. Here, let us pay attention to block, $(\hat{\phi}_b + \delta_s)|u_0| + \hat{d}$, where u_0 is the one-step-previous input and $\hat{d}$ is the estimate of the system uncertainty d^*. Although it is natural to consider that there exists a gap between the one-step-previous input u_0 and the current input u, the gap in the term above can be seen a part of d^* and be covered by $\hat{d}$ which is tuned by adaptive law (22) based on the tracking error E. Therefore, it is reasonable to regard the block as $(\hat{\phi}_b + \delta_s)|u| + \hat{d}$ in terms of the boundary-free uncertainty, and consequently, (38) becomes,

$$\begin{aligned} \dot{v} \le & -\frac{1}{2}E^T Q E - s_\phi\tilde{b}u - \delta_s|s_\phi u| \\ & -\sigma\sum_{i=1}^{n}\sum_{j=1}^{m}\tilde{w}_{aij}(\hat{w}_{aij} - w_{aij0}) + \tilde{W}_b^T\dot{\hat{W}}_b \\ & -\sigma\tilde{\Phi}_a^T(\hat{\Phi}_a - \Phi_{a0}) - \sigma\tilde{\phi}_b(\hat{\phi}_b - \phi_{b0}) - \sigma\tilde{d}\hat{d} \end{aligned} \quad (35)$$

We now consider two cases, (a) inside $\mathcal{S}$ and (b) outside $\mathcal{S}$, and show the system stability and tracking ability in both cases.

(a) In the case of inside $\mathcal{S}$:
(35) becomes

$$\begin{aligned}\dot{v} &\leq -\frac{1}{2}E^TQE - \sigma\tilde{W}_b^T(\hat{W}_b - W_{b0}) \\ &-\sigma\sum_{i=1}^{n}\sum_{j=1}^{m}\tilde{w}_{aij}(\hat{w}_{aij} - w_{aij0}) \\ &-\sigma\tilde{\Phi}_a^T(\hat{\Phi}_a - \Phi_{a0}) - \sigma\tilde{\phi}_b(\hat{\phi}_b - \phi_{b0}) - \sigma\tilde{d}\hat{d}\end{aligned} \quad (36)$$

where $\delta_s = 0$, and the upper part of adaptive law in (24) are used. Now, let's pay attention to the second right-hand term in above expression.

$$\begin{aligned}&\tilde{W}_b^T(\hat{W}_b - W_{b0}) \\ =\ &\tilde{W}_b^T\left(\tilde{W}_b + W_b^* - W_{b0}\right) \\ =\ &\tilde{W}_b^T\tilde{W}_b + \left(\hat{W}_b - W_b^*\right)^T\left(\hat{W}_b - \tilde{W}_b - W_{b0}\right) \\ =\ &\tilde{W}_b^T\tilde{W}_b + \left(\hat{W}_b - W_{b0} - W_b^* + W_{b0}\right)^T \\ &\cdot\left(\hat{W}_b - W_{b0} - \tilde{W}_b\right) \\ =\ &\tilde{W}_b^T\tilde{W}_b + \left(\hat{W}_b - W_{b0}\right)^T\left(\hat{W}_b - W_{b0}\right) \\ &-\left(\hat{W}_b - W_{b0}\right)^T\tilde{W}_b - \left(W_b^* - W_{b0}\right)^T\left(W_b^* - W_{b0}\right) \\ \geq\ &\tilde{W}_b^T\tilde{W}_b - \left(\hat{W}_b - W_{b0}\right)^T\tilde{W}_b \\ &-\left(W_b^* - W_{b0}\right)^T\left(W_b^* - W_{b0}\right)\end{aligned}$$

thus,

$$\begin{aligned}&\tilde{W}_b^T(\hat{W}_b - W_{b0}) \\ \geq\ &\frac{1}{2}\tilde{W}_b^T\tilde{W}_b - \frac{1}{2}\left(W_b^* - W_{b0}\right)^T\left(W_b^* - W_{b0}\right)\end{aligned} \quad (37)$$

Dealing with the rest terms except first term in the right-hand in (36) by the same way that (37) resulted, it leads to that

$$\begin{aligned}\dot{v} &\leq -\frac{1}{2}E^TQE - \frac{\sigma}{2}\tilde{W}_b^T\tilde{W}_b - \frac{\sigma}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\tilde{w}_{aij}^2 \\ &-\frac{\sigma}{2}\tilde{\Phi}_a^T\tilde{\Phi}_a - \frac{\sigma}{2}\tilde{\phi}_b^2 - \frac{\sigma}{2}\tilde{d}^2 + \epsilon \\ &\leq -\frac{1}{2}\lambda_{\min}(Q)E^TE - \frac{\sigma}{2}\tilde{W}_b^T\tilde{W}_b - \frac{\sigma}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\tilde{w}_{aij}^2 \\ &-\frac{\sigma}{2}\tilde{\Phi}_a^T\tilde{\Phi}_a - \frac{\sigma}{2}\tilde{\phi}_b^2 - \frac{\sigma}{2}\tilde{d}^2 + \epsilon \\ &= -\frac{1}{2}\lambda_{\min}(Q)\frac{\lambda_{\max}(P)}{\lambda_{\max}(P)}E^TE - \frac{\sigma}{2}\tilde{W}_b^T\tilde{W}_b - \frac{\sigma}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\tilde{w}_{aij}^2 \\ &-\frac{\sigma}{2}\tilde{\Phi}_a^T\tilde{\Phi}_a - \frac{\sigma}{2}\tilde{\phi}_b^2 - \frac{\sigma}{2}\tilde{d}^2 + \epsilon \\ &\leq -\frac{1}{2}\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}E^TPE - \frac{\sigma}{2}\tilde{W}_b^T\tilde{W}_b - \frac{\sigma}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\tilde{w}_{aij}^2 \\ &-\frac{\sigma}{2}\tilde{\Phi}_a^T\tilde{\Phi}_a - \frac{\sigma}{2}\tilde{\phi}_b^2 - \frac{\sigma}{2}\tilde{d}^2 + \epsilon \\ &\leq -\alpha v + \epsilon\end{aligned} \quad (38)$$

where,

$$\alpha = \min\left(\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)},\ \sigma\right) \quad (39)$$

$$\begin{aligned}\epsilon &= \frac{\sigma}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}(w_{aij}^* - w_{aij0})^2 \\ &+\frac{\sigma}{2}\left(W_b^* - W_{b0}\right)^T\left(W_b^* - W_{b0}\right) \\ &+\frac{\sigma}{2}\left(\Phi_a^* - \Phi_{a0}\right)^T\left(\Phi_a^* - \Phi_{a0}\right) \\ &+\frac{\sigma}{2}\left(\phi_b^* - \phi_{b0}\right)^2 + \frac{\sigma}{2}d^{*2}\end{aligned} \quad (40)$$

which implies,

$$\begin{aligned}v &\leq e^{-\alpha(t-t_0)}v(t_0) + \int_{t_0}^{t}e^{-\alpha(t-\tau)}\epsilon d\tau \\ &= \left(v(t_0) + \frac{\epsilon}{\alpha}\right)e^{-\alpha(t-t_0)} + \frac{\epsilon}{\alpha}\end{aligned} \quad (41)$$

Therefore, all signals in (25), which also are all signals involved in the system, are bounded. Besides, from (25) and (41), we can get that there exists $\mathcal{T}$ such that for $t \geq \mathcal{T}$, $E(t)$ satisfies

$$\|E(t)\| \leq \frac{1}{\lambda_{\min}(P)}\sqrt{\frac{2\epsilon}{\alpha}} \quad (42)$$

which implies the tracking error vector $E(t)$ is uniformly bounded, and tends to a ball centered at the origin with radius $\frac{1}{\lambda_{\min}(P)}\sqrt{\frac{2\epsilon}{\alpha}}$.

(b) In the case of outside $\mathcal{S}$:
(35) becomes

$$\begin{aligned}\dot{v} &\leq -\frac{1}{2}E^TQE - s_\phi\tilde{b}u - (|\hat{b}(X)| + \bar{b})|s_\phi u| \\ &-\sigma\sum_{i=1}^{n}\sum_{j=1}^{m}\tilde{w}_{aij}(\hat{w}_{aij} - w_{aij0}) \\ &-\sigma\tilde{\Phi}_a^T(\hat{\Phi}_a - \Phi_{a0}) - \sigma\tilde{\phi}_b(\hat{\phi}_b - \phi_{b0}) - \sigma\tilde{d}\hat{d} \\ &\leq -\frac{1}{2}E^TQE - \sigma\sum_{i=1}^{n}\sum_{j=1}^{m}\tilde{w}_{aij}(\hat{w}_{aij} - w_{aij0}) \\ &-\sigma\tilde{\Phi}_a^T(\hat{\Phi}_a - \Phi_{a0}) - \sigma\tilde{\phi}_b(\hat{\phi}_b - \phi_{b0}) - \sigma\tilde{d}\hat{d}\end{aligned} \quad (43)$$

where $\delta_s = |\hat{b}(X)| + \bar{b}$ in (19), and $\dot{\hat{W}}_b = 0$ in (24) are used. Straightforwardly with same manner as in (38), we have,

$$\dot{v} \leq \alpha v + \epsilon \quad (44)$$

where,

$$\begin{aligned}\epsilon &= \frac{\sigma}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}(w_{aij}^* - w_{aij0})^2 \\ &+\frac{\sigma}{2}\left(\Phi_a^* - \Phi_{a0}\right)^T\left(\Phi_a^* - \Phi_{a0}\right) \\ &+\frac{\sigma}{2}\left(\phi_b^* - \phi_{b0}\right)^2 + \frac{\sigma}{2}d^{*2}\end{aligned} \quad (45)$$

Clearly, (41), and (42) are satisfied as well in this case.

Remark 4 From (40), (45), and (42), we can see that the better *a priori* estimates for $\hat{\Phi}_a, \hat{\phi}_b$ and so forth, the smaller tracking error we can get. Note that such a kind of *a priori* estimates is defferent from the assumption on the *optimal fuzzy approximators.*

Remark 5 The bound of $E(t)$ also depends on the leakage constant σ and P, Q in (9), therefore, the magnitude of bound of $E(t)$ can be made arbitrarily small by adjusting the parameters σ, P, and Q.

The results achieved in this paper can be summarized in a theorem as follows:

Theorem 2 *If the plant (1) subject to the Assumption in section 1 is controlled by (17)-(19) with the adaptive law (20)-(24), then tracking error, between the states of the considered system and its reference model (3), will be uniformly bounded, also the bound can be made arbitrarily small by choosing appropriately control parameters, while maintaining all signals in the system asymptotically stable.*

4 Simulation

In order to clarify the proposed design of procedure, we apply the adaptive fuzzy controller developed in the previous section to control the following nonlinear system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \left(\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} \right) u + \begin{bmatrix} 0 \\ d \end{bmatrix} \quad (46)$$

where,

$$a = \frac{1 - \exp(-x_2)}{1 + \exp(-x_2)}$$

$$b = 0.5\sin(x_1), \quad d = 0.5\sin(t)$$

Its reference model is given by

$$\begin{bmatrix} \dot{x}_{1m} \\ \dot{x}_{2m} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} x_{1m} \\ x_{2m} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin(t) \quad (47)$$

For the reference model above, and a given positive definite matrix,

$$Q = \begin{bmatrix} 16 & 0 \\ 0 & 4 \end{bmatrix} \quad (48)$$

there is a positive definite matrix that is the solution of (9),

$$P = \begin{bmatrix} 12 & 2 \\ 2 & 2 \end{bmatrix} \quad (49)$$

Apparently, in the plant there are known parts $A_0 = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, b_0 = 1$, and unknown parts $A = \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix}$ and b. For the known parts, they are used in the controller (17) directly, while unknown functions a, b related with the unknown parts will be treated using the fuzzy approximator as mentioned in section 3. Since the unknown functions a, and b are the functions with respect to x_2, and x_1, respectively, the variables in the antecedent of fuzzy approximators are x_2, and x_1, respectively as well:

$$R_j : IF\ x_1\ is\ A_j\ THEN\ b\ is\ w_{1j}$$

or

$$R_j : IF\ x_2\ is\ A_j\ THEN\ a\ is\ w_{2j}$$

where j is rule's number; A_j is a fuzzy set, and $w_{\cdot j}$ is a singleton value. In this simulation, we adopt seven fuzzy sets over the domains of x_1, and x_2. Consequently, it yields seven fuzzy rules at most, where each fuzzy rule R_j $(j = 1, 2, \ldots, 7)$ has a consequent $w_{\cdot j}$. Furthermore, seven fuzzy sets are given as in Fig.1, $w_{\cdot j}$ is initially assigned 1 and to be tuned by adaptive law.

Control input is determined by (17-19) with adaptive

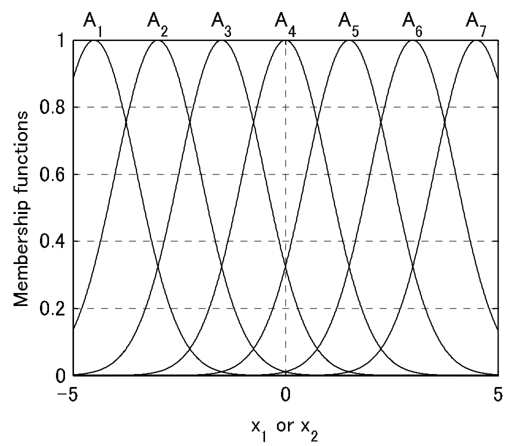

Figure 1: Membership functions in antecedent

law (20)-(24), where $\sigma = 0.1$, and all the prior estimates for Φ_a, ϕ_b, $\mathbf{W}_a$, and W_b are taken as 0 in order to show that we have no any such a prior knowledge on the system. For the upper bound $\bar{b}$, in this simulation fuzzy approximator $b^*(X)$ is used to deal with unknown function $b = 0.5\sin(x_1)$, therefore the upper bound is taken as $\bar{b} = 0.5$. In additon, the initial states for the plant and the reference model are taken as $x_1(0) = 3, x_2(0) = 2, x_{1m}(0) = 1$, $x_{2m}(0) = -1$, and all initial values for the adaptive law are taken as 0.5.

Simulation results are shown in Figs.2-4. Figs.2-3 depicted the evolutions of x_1, x_2, and their desired states from the reference model, in which good tracking performance is observed in spite of the fact that there is a sin-curved uncertainty (disturbance) in the plant. The amount of control effort required to achieve the above level of the performance is illustrated in Fig.4, where the boundary layer is taken $\phi = 0.01$.

5 Conclusion

In this paper, we proposed a fuzzy controller for a class of reference model adaptive control systems with

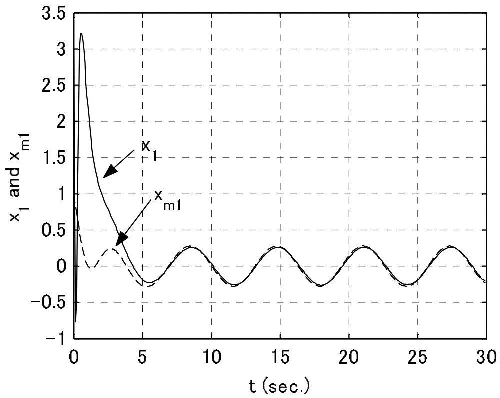

Figure 2: Evolutions of states x_1, and x_{m1}

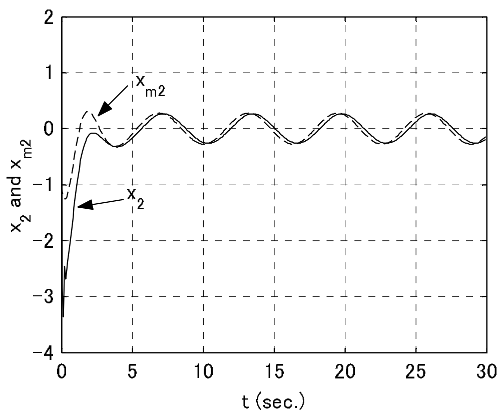

Figure 3: Evolutions of states x_2, and x_{m2}

uncertainties. Shortly, the main issues we considered here were: (1) when using the fuzzy approximator to deal with some unknown functions in the plant to be controlled, the known parts, if any, in the functions are utilized to the maximum even it was a tough task to deal with the control gain in such a way; (2) regarding the unknown parts to be approximated, the prior knowledge, if any, about the parts also should be used as much as possible, since, as mentioned in remark 4, the tracking error is directly related with such knowledge.

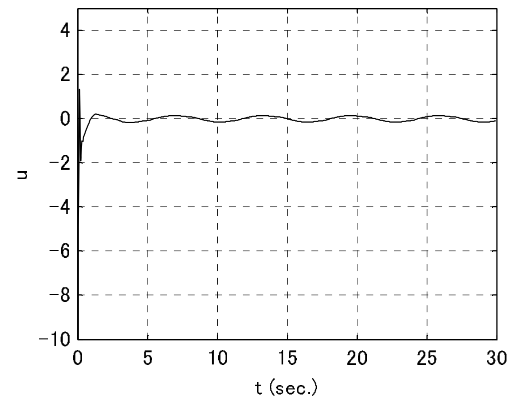

Figure 4: Amount of control input u

References:

[1] K. Tanaka, and M. Sugeno, "Stability analysis and design of fuzzy control systems," *Fuzzy Sets and Systems,* vol. 45, pp. 135-156, 1992.

[2] L.-X. Wang, "Stable adaptive fuzzy control of nonlinear systems," *IEEE trans. Fuzzy Systems,* vol. 1, no. 2, pp. 146-155, 1993.

[3] C.-Y. Su, and Y. Stepanenko, "Adaptive control of a class of nonlinear systems with fuzzy logic," *IEEE trans. Fuzzy Systems,* vol. 2, no.4, pp. 285-294, 1994.

[4] M. M. Polycarpou, "Stable adaptive neural control scheme for nonlinear systems," *IEEE Trans. on Automatic Control*, vol. 41, no. 3, pp. 447-451, 1996.

[5] M. M. Polycarpou, and M. J. Mears, "Stable adaptive tracking of uncertain systems using nonlinear parametrized on-line approximators," *Int. J. Control*, vol. 73, no. 3, pp. 363-384, 1998.

[6] J.T. Spooner, and K.M. Passino, "Decentralized adaptive control of nonlinear systems using Radial basis neural networks," *IEEE Trans. on Auto. Control,* vol.44, no.11, pp.2050-2057, 1999

[7] H. Han, C.-Y. Su, and Y. Stepanenko, "Adaptive control of a class of nonlinear systems with nonlinearly parameterized fuzzy approximators," *IEEE Trans. on Fuzzy Systems,* vol. 9, no. 2, pp. 315-323, 2001.

[8] C.-H. Wang, H.-L. Liu, and T.-C. Lin, "Direct adaptive fuzzy-neural control with state observer and supervisory controller for unknown nonlinear dynamical systems," *IEEE Trans. on Fuzzy Systems,* vol. 10, no. 1, pp. 39-49, 2002.

[9] N. Golea, A. Golea, and K. Benmahammed, "Fuzzy model reference adaptive control," *IEEE Trans. on Fuzzy Systems,* vol. 10, no. 4, pp. 436-444, 2002.

[10] H.-X. Li, and S. Tong, "A hybrid adaptive fuzzy control for a class of nonlinear MIMO systems," *IEEE Trans. on Fuzzy Systems,* vol. 11, no. 1, pp. 24-34, 2003

[11] H. Han, and C.-Y. Su, "Robust fuzzy control of nonlinear systems using shape-adaptive radial basis functions," *Fuzzy Sets and Systems,* vol. 125, no. 1, pp. 23-38, 2002

[12] P.A. Ioannou, and J. Sun, *Robust Adaptive Control,* Prentice-Hall PTR, 1996.

[13] J.-J. Slotine, "Sliding controller design for non-linear systems," *Int. J. Control,* vol. 40, no. 2, pp.421-434, 1984.

[14] L.-X. Wang, and J. M. Mendel, "Fuzzy basis functions, universal approximation, and orthogonal least-squares learning," *IEEE Trans. on Neural Networks*, vol. 3, pp. 807-813, 1992.

[15] R. M. Sanner, and J.-J. Slotine, "Gaussian network for direct adaptive control," *IEEE Trans. on Neural Networks,* vol. 3, no. 6, pp.837-863, 1992.

[16] S. Sastry, and M. Bodson, *Adaptive Control, Stability, Convergence, and Robustness,* Prentice Hall, 1989.