Difference approximation of Hamilton Jacobi equation. Convergence

ANDRZEJ NOWAKOWSKI, IWONA NOWAKOWSKA

Faculty of Mathematics, Universiti of Lodz
Banacha 22, 90-232 Łódź
POLAND
Academy of Humanities and Economics in Lodz
Rewolucji 1905r 52, 90-222 Lodz
POLAND

Abstract: The aim of the paper is to describe numerical approximation of the difference Hamilton-Jacobi inequality $-2\varepsilon \leq \frac{S_\varepsilon(t+h)-S_\varepsilon(t,x)}{h} + H(t,x,-S_{\varepsilon x}(t,x)) \leq \varepsilon$, and to prove its convergence. We find, by numerical construction, a function $S_\varepsilon(t,x)$ which satisfies the above inequality. The method applied in the paper bases on the constructions described in [1] for Bolza problem and significently extended in [3].

Key Words: Hamilton-Jacobi equation, difference equation, approximate solution, convrgence of approximation.

1 Introduction

Let $S(t,x)$ be a $C^1(T)$, $T \subset R^{1+n}$, function satisfying Hamilton-Jacobi equation

$$S_t(t,x) + H(t,x,-S_x(t,x)) = 0, \tag{1}$$

$$(t,x) \in T = [0,b] \times A,\ S(b,x) = l(x),$$

where A is a compact set in R^n with nonempty interior and $l(x)$ is a C^1function in A. That equation is fundamental in mechanics and becomes very useful in system theory. If $S(t,x)$ is treated as a value function of some optimal control or variational problem then its approximate $S_\varepsilon(t,x)$ must satisfy approximate Hamilton-Jacobi inequality (the verification inequality of the dynamic programming) see e.g. [2] or [1] i.e.

$$-\varepsilon \leq S_{\varepsilon t}(t,x) + H(t,x,-S_{\varepsilon x}(t,x)) \leq 0, \tag{2}$$

$(t,x) \in T$, and the boundary condition $l(x) \leq S(b,x) \leq l(x) + \varepsilon(b-a), (b,x) \in T$. We can approximate the derivative $S_{\varepsilon t}(t,x)$ by its difference $\frac{S_\varepsilon(t+h,x)-S_\varepsilon(t,x)}{h}$, let us assume, uniformly in T. Then we should still have the verification inequality of the dynamic programming

$$-2\varepsilon \leq \frac{S_\varepsilon(t+h)-S_\varepsilon(t,x)}{h} \tag{3}$$

$$+H(t,x,-S_{\varepsilon x}(t,x)) \leq \varepsilon,$$

$$(t,x),\ (t+h,x) \in T\ .$$

In this paper we confine ourselves to the case when Hamiltonian is of the form:

$$\begin{aligned} & H(t,x,w(t,x)) \\ = \ & \min_{u \in U} \left\{ \frac{\partial w}{\partial x}(t,x) f(t,x,u) + L(t,x,u) \right\} \end{aligned}$$

where $(t,x,u) \longrightarrow f(t,x,u)$ and $(t,x,u) \rightarrow L(t,x,u)$ are Lipschitz continuous functions defined in $T \times U$ with values in R^n and R, respectively, $U \subset R^m$ is compact. That means that

$S_\varepsilon(t,x)$ from (2) is now an $\varepsilon-$value function for suitable control problem with Lagrangian L and velocity f of the state $x(t)$.

The aim of the paper is to describe numerical approximation of the difference Hamilton-Jacobi inequality (3) and to prove its convergence i.e. we want to find by numerical construction a function $S_\varepsilon(t,x)$ in T which satisfies (3). The method applied in this paper bases on the constructions described in [1] for Bolza problem and significently extended in [3]. Here we make some further extensions and simplification of both papers according to the problem considered in this paper.

2 Construction of the approximation

We begin the construction of the $\varepsilon-$value function which should satisfy (3) by choosing some arbitrary function $(t,x) \to w(t,x)$ of class $C^2(T)$, that satisfies the boundary condition: $w(b,x) = l(x)$, $(b,x) \in T$.

We define on T a function $(t,x) \to F(t,x)$ that corresponds to the right-hand side of the Hamilton-Jacobi difference equation:

$$F(t,x) := \frac{w(t+h,x) - w(t,x)}{h} \qquad (4)$$

$$+\min_{u\in U}\left\{\frac{\partial w}{\partial x}(t,x)f(t,x,u) + L(t,x,u)\right\}.$$

The function $(t,x) \to F(t,x)$ is a Lipschitz function on T, as U is compact and the functions in bracket $\{\}$ are Lipschitz continuous. Since T is a compact set, the function $F(\cdot,\cdot)$ is bounded in T from below and above by κ_d and κ_g, respectively:

$$\kappa_d \le F(t,x) \le \kappa_g \text{ for all } (t,x) \in T. \qquad (5)$$

Generally, function $F(\cdot,\cdot)$ has values in T of different signs, therefore it does not satisfy the verification inequality of the dynamic programming (3), which requires that $F(\cdot,\cdot)$ has non-positive values greater than -2ε and less than ε on whole T. In order to find a function that satisfies the verification inequality of the dynamic programming we definea family of functions $(t,x) \to F_1^k(t,x)$, $k \in \mathbb{N}$, in T. These functions will satisfy for all $k > k_\varepsilon$ the inequality given in (3), where $k_\varepsilon \in \mathbb{N}$ is a number that depends on chosen ε, such that $k_\varepsilon \to \infty$ for $\varepsilon \to 0$. The function $F_1^k(\cdot,\cdot)$ for every k is described by the following formula and the construction of $(t,x) \to w_1^k(t,x)$, $k \in \mathbb{N}$ is described below:

$$F_1^k(t,x) := \frac{w_1^k(t+h,x) - w_1^k(t,x)}{h}$$

$$+\min_{u\in U}\left\{\frac{\partial w_1^k}{\partial x}(t,x)f(t,x,u) + L(t,x,u)\right\}$$

$$\text{for every } (t,x) \in T. \qquad (6)$$

We begin the construction of $w_1^k(\cdot,\cdot)$ from defining its domain. Let us divide the interval $[\kappa_d,\kappa_g] \subset \mathbb{R}$, being the image of the set T in the mapping $(t,x) \to F(t,x)$, creating k subinterval $[y_i, y_{i+1}]$, $i \in \{1,\ldots,k\}$, such that: $\kappa_d = y_1 < y_2 < \ldots < y_{k+1} = \kappa_g$, and that for all $i \in \{1,\ldots,k\}$ we have $|y_{i+1} - y_i| = \frac{1}{k}|\kappa_g - \kappa_d|$. Obviously it is the equipartition of the interval $[\kappa_d,\kappa_g]$. Let us introduce the following symbol: $\eta_k := \frac{1}{k}|\kappa_g - \kappa_d|$.

Now we divide set T into following subsets P_j^k, $j \in \{1,\ldots,k\}$:

$$P_1^k \ : \ = \{(t,x) \in T : y_1 \le F(t,x) \le y_2\} \qquad (7)$$

$$P_j^k \ : \ = \{(t,x) \in T : y_j < F(t,x) \le y_{j+1}\}, \qquad (8)$$

$$j \ \in \ \{2,\ldots,k\}, \qquad (9)$$

The sets P_j^k, $j \in \{1,\ldots,k\}$ constitute a covering of the set T, i.e. for every $i,j \in \{1,\ldots,k\}$, $i \ne j$, $P_i^k \cap P_j^k = \varnothing$, and $\bigcup_{j=1}^k P_j^k = T$.

Now define auxiliary functions $(t,x) \to w_{1,j}^k(t,x)$ and $(t,x) \to F_{1,j}(t,x)$ on sets P_j^k, $j \in \{1,\ldots,k\}$ as follows:

$$w_{1,j}^k(t,x) := w(t,x) + y_{j+1}(b-t), (t,x) \in P_j^k, \qquad (10)$$

$$F_{1,j}^k(t,x) := \frac{w_{1,j}^k(t+h,x) - w_{1,j}^k(t,x)}{h}$$

$$+\min_{u\in U}\left\{\frac{\partial w_{1,j}^k}{\partial x}(t,x)f(t,x,u) + L(t,x,u)\right\},$$

$$(t,x) \in P_j^k. \qquad (11)$$

By simple calculation we obtain that:

$$F_{1,j}^k(t,x) = F(t,x) - y_{j+1}, \ (t,x) \in P_j^k, \qquad (12)$$

which means that the following inequality holds:

$$-\eta_k \le F_{1,j}^k(t,x) \le 0, \ (t,x) \in P_j^k, \ j \in \{1,\ldots,k\}. \qquad (13)$$

It is easily to notice that for some fixed $\varepsilon > 0$ we can always choose such k_ε, that for every $m > k_\varepsilon$ we have $\varepsilon \leq F^m_{1,j}(t,x) \leq 0$.

We define the function $w^k_1(\cdot,\cdot)$ (for fixed k) in $T = \bigcup_{j=1}^k P^k_j$ as follows:

$$w^k_1(t,x) \quad : \quad = w^k_{1,j}(t,x) \tag{14}$$
$$\text{for } (t,x) \quad \in \quad P^k_j,\ j \in \{1,\dots,k\}. \tag{15}$$

Obviously for every $k > k_\varepsilon$ the function $w^k_1(\cdot,\cdot)$ satisfies the inequality of the verification theorem of the dynamic programming for fixed $\varepsilon > 0$, and satisfies the boundary condition of this theorem, yet it is not a function of class $C^1(T)$ (probably it is even a non-continuous function), and thus it is not an $\varepsilon-$value function. In order to satisfy the assumptions of the verification theorem we have to smoothen the function $w^k_1(\cdot,\cdot)$ by convoluting it with a function of class $C^\infty(\mathbb{R}^{n+1})$ having compact support.

From now on we assume that k (the number of sets P^k_j) is a fixed natural number, $j \in \{1,\dots,k\}$, and $\beta > 0$ is some real number.

The function $\rho_\beta : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ of class $C^\infty(\mathbb{R}^{n+1})$ having compact support, where $\beta \in \mathbb{R}_+$ is defined as follows:

Let $\rho_1 : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a function of class $C^\infty(\mathbb{R}^{n+1})$ having compact support, such that $\int_{\mathbb{R}^{n+1}} \rho_1(t,x)dtdx = 1$ and *supp* $\rho_1 \subset B_1(\mathbb{R}^{n+1})$, where *supp* denotes the support, and $B_\tau(\mathbb{R}^{n+1})$ for any $\tau \in \mathbb{R}$ is a ball in $\mathbb{R}^{n+1}$ with center in 0 having radius τ. Obviously $\rho_\beta(t,x) := \frac{1}{\beta^{n+1}}\rho_1(\frac{t}{\beta},\frac{x}{\beta})$. It is easy to see that such function $\rho_\beta(\cdot,\cdot)$ is infinitely smooth function having compact support and *supp* $\rho_\beta \subset B_\beta(\mathbb{R}^{n+1})$ and $\int_{B_\beta(\mathbb{R}^{n+1})} \rho_\beta(t,x)dtdx = \int_{\mathbb{R}^{n+1}} \rho_\beta(t,x)dtdx = 1$.

Let now define for each $\beta \in \mathbb{R}_+$ a new function $(t,x) \to w^{k,\beta}_2(t,x)$:

$$w^{k,\beta}_2(t,x) := (w^k_1 * \rho_\beta)(t,x) \tag{16}$$

where the "$*$" denotes convolution.

By known theorem we have for every k and β the function $w^{k,\beta}_2(\cdot,\cdot)$ is of class $C^\infty(T)$, what means that the corresponding function $(t,x) \to F^{k,\beta}_2(t,x)$, having following definition:

$$F^{k,\beta}_2(t,x) := \frac{w^{k,\beta}_2(t+h,x) - w^{k,\beta}_2(t,x)}{h} \tag{17}$$
$$+\min_{u\in U}\left\{\frac{\partial w^{k,\beta}_2}{\partial x}(t,x)f(t,x,u) + L(t,x,u)\right\}$$

is Lipschitz continuous in T.

3 The proof of the convergence

In this section we will try to evaluate function $F^{k,\beta}_2(\cdot,\cdot)$ to prove that $w^{k,\beta}_2(\cdot,\cdot)$ is a function which we are looking for.

For every given, fixed k and for every $i \in \mathbb{N}$ there exists such real $\hat\beta^{k,i} > 0$, that for every $0 < \beta \leq \hat\beta^{k,i}$ and for all $(t,x) \in T$ the following inequality is satisfied:

$$\left|\frac{w^{k,\beta}_2(t+h,x) - w^{k,\beta}_2(t,x)}{h} - \frac{w^k_1(t+h,x) - w^k_1(t,x)}{h}\right| < (1+\frac{1}{i})\eta_k. \tag{19}$$

Let us take any $(t,x) \in T$. Then for a certain $m \in \{1,\dots,k\}$, $(t,x) \in P^k_m$ and may be another $m' \in \{1,\dots,k\}$ $(t+h,x) \in P^k_{m'}$. Since $F(\cdot,\cdot)$ is uniformly continuous on T, therefore there is $\hat\beta^k_1$ such that for $(s,y) \in B_{\hat\beta^k_1}(\mathbb{R}^{n+1})$: $|F(t-s,x-y) - F(t,x)| < \frac{1}{2}\eta_k$ and $|F(t+h-s,x-y) - F(t+h,x)| < \frac{1}{2}\eta_k$. Therefore either $\forall_{(s,y)\in B_{\hat\beta^k_1}(\mathbb{R}^{n+1})} (t-s,x-y) \in P^k_m \cup P^k_{m-1}$ or $\forall_{(s,y)\in B_{\hat\beta^k_1}(\mathbb{R}^{n+1})} (t-s,x-y) \in P^k_m \cup P^k_{m+1}$ and either $\forall_{(s,y)\in B_{\hat\beta^k_1}(\mathbb{R}^{n+1})} (t+h-s,x-y) \in P^k_{m'} \cup P^k_{m'-1}$ or $\forall_{(s,y)\in B_{\hat\beta^k_1}(\mathbb{R}^{n+1})} (t+h-s,x-y) \in P^k_{m'} \cup P^k_{m'+1}$. Assume that the first alternative holds in both cases. The proofs of the other cases are similar.

Let us put:

$$D^m_\beta := \left\{(s,y) \in B_\beta(\mathbb{R}^{n+1}) : (t-s,x-y) \in P^k_m\right\},$$

$$D^{m-1}_\beta := \left\{(s,y) \in B_\beta(\mathbb{R}^{n+1}) : (t-s,x-y) \in P^k_{m-1}\right\}$$

and

$$D^{m'}_\beta \quad : \quad = \left\{(s,y) \in B_\beta(\mathbb{R}^{n+1}) : (t+h-s,x-y) \in P^k_{m'}\right\},$$

$$D^{m'-1}_\beta \quad : \quad = \left\{(s,y) \in B_\beta(\mathbb{R}^{n+1}) : (t+h-s,x-y) \in P^k_{m'-1}\right\}.$$

We have for $0 < \beta < \hat{\beta}_1^k$: $B_\beta(\mathbb{R}^{n+1}) = D_\beta^m \cup D_\beta^{m-1}$ and $D_\beta^m \cap D_\beta^{m-1} = \varnothing$ and similarly $B_\beta(\mathbb{R}^{n+1}) = D_\beta^{m'} \cup D_\beta^{m'-1}$ and $D_\beta^{m'} \cap D_\beta^{m'-1} = \varnothing$.

Having all above we are able to calculate:

$$\left| \frac{w_2^{k,\beta}(t+h,x) - w_2^{k,\beta}(t,x)}{h} - \frac{w_1^k(t+h,x) - w_1^k(t,x)}{h} \right| \tag{20}$$

$$= \frac{1}{h} \left| \int\limits_{B_\beta(\mathbb{R}^{n+1})} \Big(w_2^{k,\beta}(t+h-s,x-y) - w_2^{k,\beta}(t-s,x-y) \Big) \rho_\beta(s,y) ds dy - \Big(w_1^k(t+h,x) - w_1^k(t,x) \Big) \right|$$

$$\le \tfrac{1}{h} \Big| \int\limits_{D_\beta^m} (w_1^k(t,x) - w_{1,m}^k(t-s,x-y))\rho_\beta(s,y) ds dy \Big| + \int\limits_{D_\beta^{m-1}} (w_1^k(t,x) - w_{1,m-1}^k(t-s,x-y))\rho_\beta(s,y) ds dy$$

$$+ \int\limits_{D_\beta^{m'}} \Big(w_{1,m'}^k(t+h-s,x-y) - w_1^k(t+h,x)\rho_\beta(s,y) ds dy \Big) + \int\limits_{D_\beta^{m'-1}} \Big(w_{1,m'-1}^k(t+h-s,x-y) - w_1^k(t+h,x)\rho_\beta(s,y) ds dy \Big) \Big|$$

$$\le \frac{1}{h} \Big| \int\limits_{B_\beta(\mathbb{R}^{n+1})} (w(t,x) - w(t-s,x-y))\rho_\beta(s,y) ds dy + \int\limits_{B_\beta(\mathbb{R}^{n+1})} (w(t+h-s,x-y) - w(t+h,x)\rho_\beta(s,y) ds dy) \Big|$$

$$+ \int\limits_{B_\beta(\mathbb{R}^{n+1})} |y_{m+1} - y_m| \, \rho_\beta(s,y) ds dy$$

$$\le \frac{1}{h} \sup_{(s,y)\in B_\beta(\mathbb{R}^{n+1})} \{ |w(t-s,x-y) - w(t,x) + w(t+h-s,x-y) - w(t+h,x)| \} + |y_{m+1} - y_m| \le 2M_w^t \sqrt{n+1}\beta + \eta_k,$$

where M_w^t is Lipschitz constant for $(t,x) \to \frac{w(t+h,x)-w(t,x)}{h}$ in T. Thus there is $0 < \hat{\beta}^{k,i} \le \hat{\beta}_1^k$, such that $2M_w^t\sqrt{n+1}\beta < \frac{1}{i}\eta_k$ and in consequence:

$$\left| \frac{w_2^{k,\beta}(t+h,x) - w_2^{k,\beta}(t,x)}{h} - \frac{w_1^k(t+h,x) - w_1^k(t,x)}{h} \right| \tag{21}$$

$$< \frac{1}{i}\eta_k + \eta_k, \ (t,x) \in T,$$

and so the proof is completed.

Similarly we prove the estimation for the derivative of $w_2^{k,\beta}(t,x) - w_1^k(t,x)$ with respect to x, i.e. the following

For a given, fixed k and for every $i \in \mathbb{N}$ there exists such $\check{\beta}^{k,i} > 0$, that for every $0 < \beta \le \check{\beta}^{k,i}$ and for all $(t,x) \in T$ the following inequality holds:

$$\left| \frac{\partial}{\partial x} w_2^{k,\beta}(t,x) - \frac{\partial}{\partial x} w_1^k(t,x) \right| < \frac{1}{i}\eta_k. \tag{22}$$

For fixed k and for all $(t,x) \in T$:

$$\lim_{\beta\to 0} \frac{\partial}{\partial x} w_2^{k,\beta}(t,x) = \frac{\partial}{\partial x} w_1^k(t,x) \tag{23}$$

and the convergence is uniform.

To simplify the notation we will define two auxiliary functions on $T \times U$, $\beta > 0$:

$$g_1^k(t,x,u) \ : \ = \frac{\partial}{\partial x} w_1^k(t,x) f(t,x,u) + L(t,x,u), \tag{24}$$

$$g_2^{k,\beta}(t,x,u) \tag{25}$$

$$: \ = \frac{\partial}{\partial x} w_2^{k,\beta}(t,x) f(t,x,u) + L(t,x,u). \tag{26}$$

In order to prove the convergnce of $w_2^{k,\beta}(t,x)$ we need the following convergence lemma

For fixed k and for all $(t,x,u) \in T \times U$:

$$\lim_{\beta\to 0} g_2^{k,\beta}(t,x,u) = g_1^k(t,x,u) \tag{27}$$

and the convergence is uniform.

It is enough to show that for any $\epsilon > 0$ there exists $\tilde{\beta}$ such that for $0 < \beta \leq \tilde{\beta}$ and $(t,x,u) \in T \times U$ we have:

$$\left| g_2^{k,\beta}(t,x,u) - g_1^k(t,x,u) \right| < \epsilon \tag{28}$$

By the above lemma for $i \in \mathbb{N}$ there is such $\breve{\beta}^{k,i} > 0$, that for $0 < \beta \leq \breve{\beta}^{k,i}$ and $(t,x,u) \in T \times U$:

$$\left| g_2^{k,\beta}(t,x,u) - g_1^k(t,x,u) \right|$$

$$\begin{aligned} &= \left| \frac{\partial}{\partial x} w_2^{k,\beta}(t,x) - \frac{\partial}{\partial x} w_1^k(t,x) \right| |f(t,x,u)| \\ &< \frac{1}{i} \eta_k M_f, \end{aligned} \tag{29}$$

where M_f is boundedness of $|f(t,x,u)|$ on $T \times U$. Hence, taking $\tilde{\beta} := \breve{\beta}^{k,i}$, $i \in \mathbb{N}$ such that $\frac{1}{i}\eta_k M_f \leq \epsilon$ we conclude the assertion of the lemma.

Let us introduce some additional symbols ($\beta > 0$):

$$\begin{aligned} p_1^k(t,x) &= \min_{u \in U} g_1^k(t,x,u) = g_1^k(t,x,u_1^k(t,x)) \\ p_2^{k,\beta}(t,x) &= \min_{u \in U} g_2^{k,\beta}(t,x,u) \\ &= g_2^{k,\beta}(t,x,u_2^{k,\beta}(t,x)), \end{aligned} \tag{30}$$

where $u_1^k(t,x)$, $u_2^{k,\beta}(t,x)$ are such values of controls that minimize the respective functions g_1^k and $g_2^{k,\beta}$ at point (t,x).

For fixed k and for all $(t,x) \in T$:

$$\lim_{\beta \to 0} p_2^{k,\beta}(t,x) = p_1^k(t,x) \tag{31}$$

and this convergence is uniform.

Let us devide T on two sets: $Z' := \{(t,x) \in T : p_2^{k,\beta}(t,x) \geq p_1^k(t,x)\}$ and $Z'' := \{(t,x) \in T : p_2^{k,\beta}(t,x) < p_1^k(t,x)\}$. It is clear that $Z' \cup Z'' = T$ and $Z' \cap Z'' = \varnothing$.

Let $\tilde{\epsilon} > 0$ and $\beta_{\tilde{\epsilon}} > 0$ be such that accordingly to the above lemma for $0 < \beta < \beta_{\tilde{\epsilon}}$ and all $(t,x,u) \in T \times U$:

$$\left| g_2^{k,\beta}(t,x,u) - g_1^k(t,x,u) \right| < \tilde{\epsilon}. \tag{32}$$

Take $(t,x) \in Z'$ then:

$$0 \leq \left| p_2^{k,\beta}(t,x) - p_1^k(t,x) \right|$$

$$\begin{aligned} &= p_2^{k,\beta}(t,x) - p_1^k(t,x) = \\ &= g_2^{k,\beta}(t,x,u_2^{k,\beta}(t,x)) - g_1^k(t,x,u_1^k(t,x)) \\ &\leq g_2^{k,\beta}(t,x,u_1^k(t,x)) - g_1^k(t,x,u_1^k(t,x)) \\ &\leq \left| g_2^{k,\beta}(t,x,u_1^k(t,x)) - g_1^k(t,x,u_1^k(t,x)) \right| < \tilde{\epsilon} \end{aligned} \tag{33}$$

If $(t,x) \in Z''$ then:

$$0 \leq \left| p_2^{k,\beta}(t,x) - p_1^k(t,x) \right|$$

$$\begin{aligned} &= p_1^k(t,x) - p_2^{k,\beta}(t,x) = \\ &= g_1^k(t,x,u_1^k(t,x)) - g_2^{k,\beta}(t,x,u_2^{k,\beta}(t,x)) \quad (34) \\ &\leq g_1^k(t,x,u_2^{k,\beta}(t,x)) - g_2^{k,\beta}(t,x,u_2^{k,\beta}(t,x)) \\ &\leq \Big| g_2^{k,\beta}(t,x,u_2^{k,\beta}(t,x)) \quad (35) \\ &\qquad - g_1^k(t,x,u_2^{k,\beta}(t,x)) \Big| < \tilde{\epsilon} \end{aligned}$$

Thus for all $(t,x) \in T$ the assertion of the lemma holds.

For fixed k and any $i \in \mathbb{N}$ there is $\tilde{\beta}^{k,i} > 0$, such that for each $0 < \beta \leq \tilde{\beta}^{k,i}$ and $(t,x,u) \in T \times U$ the following inequality is satsfied:

$$\left| p_2^{k,\beta}(t,x) - p_1^k(t,x) \right| < \frac{1}{i}\eta_k. \tag{36}$$

Since $p_2^{k,\beta}(t,x)$ is uniformly convergent to $p_1^k(t,x)$ with $\beta \to 0$ on $T \times U$ thus the assertion is a direct consequence of the definition of the limit.

We are now ready to give the theorem on the convergence of our approximation.

For a given, fixed k and for any $i \in \mathbb{N}$ there exists $\bar{\beta}^{k,i} > 0$, that for every $0 < \beta \leq \bar{\beta}^{k,i}$ and for all $(t,x) \in T$ the following inequality holds:

$$\left| F_2^{k,\beta}(t,x) - F_1^k(t,x) \right| < \frac{2}{i}\eta_k + \eta_k. \tag{37}$$

We have the following estimation, for all $(t,x) \in T$:

$$\left| F_2^{k,\beta}(t,x) - F_1^k(t,x) \right|$$

$$\begin{aligned} &= \Big| \frac{\partial}{\partial t} w_2^{k,\beta}(t,x) + p_2^{k,\beta}(t,x) \\ &\qquad - D_w^{t,k}(t,x) - p_1^k(t,x) \Big| \quad (38) \\ &\leq \left| \frac{\partial}{\partial t} w_2^{k,\beta}(t,x) - D_w^{t,k}(t,x) \right| \\ &\qquad + \left| p_2^{k,\beta}(t,x) - p_1^k(t,x) \right| \quad (39) \\ &< \frac{1}{i}\eta_k + \eta_k + \frac{1}{i}\eta_k = \frac{2}{i}\eta_k + \eta_k, \end{aligned}$$

where $0 < \beta \leq \bar{\beta}^{k,i} := \min(\hat{\beta}^{k,i}, \tilde{\beta}^{k,i})$. Hence for all $(t,x) \in T$ and $i = 2, 3, ...$ and $0 < \beta < \bar{\beta}^{k,i}$ the function $F_2^{k,\beta}(\cdot,\cdot)$ may be estimated as:

$$\begin{aligned} -3\eta_k &\leq -\frac{2}{i}\eta_k - \eta_k - \eta_k \leq F_2^{k,\beta}(t,x) \\ &\leq \frac{2}{i}\eta_k + \eta_k \leq 2\eta_k. \quad (40) \end{aligned}$$

The function $w_2^{k,\beta}(\cdot,\cdot)$, is of $C^1(T)$, therefore if our division of $[\kappa_d, \ \kappa_g]$, is such that $\eta_k = \frac{1}{k}|\kappa_g - \kappa_d|$ is less than $\varepsilon/2$ then $w_2^{k,\beta}(\cdot,\cdot)$ is our $\varepsilon-$value function in T, according to the verification inequality of the dynamic programming .

4 Conclusion

The paper gives the construction of an approximate solution to the Hamilton-Jacobi difference equation (3). What is here the most interesting that it is done in two steps only: first we define the function $w_1^k(t,x)$ as an effect of special division of the set T (this function is not a smooth function), next we simply smoothen it (in suitable way) and then it is what we are looking for i.e. an $\varepsilon-$approximate solution to the Hamilton-Jacobi difference inequality (3).

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