# State Estimation for Linear Impulsive Control Systems*

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*Abstract:* - The paper deals with the state estimation problem for impulsive control system described by linear differential equations containing impulsive terms (or measures). Models of this kind arise in applied areas ranging from space navigation to investment problems as well as ecological management. The aim of the paper is to find the external set-valued estimates of the reachable sets of impulsive control systems with special ellipsoidal constrains on the admissible values of control functions and on the initial state vectors. Basing on the techniques of so-called ellipsoidal calculus we give a new state estimation approach that uses the impulsive structure of the control problem and is based on external ellipsoidal approximation of a convex union of ellipsoids. The examples of construction of such external state estimates for linear impulsive control systems are given.

*Keywords:* - Impulsive control, reachable sets, estimation.

## 1 Introduction

In this paper the impulsive control problem for a dynamic systems with unknown but bounded initial states is studied. Such problems arise from mathematical models of dynamical and physical systems for which we have an incomplete description or a loose mode of time dependence of their generalized coordinates [1, 2, 3, 4, 5, 6, 7, 8]. The topics of this paper come from the theory of systems with unknown, but bounded uncertainties (the case of the so-called "set-membership" description of uncertainties). The numerical simulation schemes developed for such problems require techniques of set–valued analysis, particularly its constructive methods — ellipsoidal or box–valued calculus [4, 3, 9].

There is a long list of publications devoted to impulsive control optimisation problems, among them we mention here only the results related to the present investigation [10, 11, 12, 13, 14, 15].

In this paper we apply the well known results of the theory [3, 4] of ellipsoidal estimating of the states of dynamical control systems with classical (measurable) controls and construct the estimation algorithms that allow to find set-valued bounds for the reachable sets of impulsive control problem under uncertainty.

We study the problem under a special restriction on control functions defined by a given generalized "ellipsoid" in the space of functions of bounded variations. In particular, under such restriction vectors of impulsive jumps of admissible controls have to belong to a given finite-dimensional ellipsoid.

We introduce here state estimation algorithms based on the properties and on the special structure of solutions of differential systems with impulsive controls, in particular we construct the ellipsoidal estimates for a convex envelope of the union of related ellipsoids of a finite dimensional space. The examples of external ellipsoidal estimates of reachable sets of linear impulsive control systems are given also.

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## 2 Problem Formulation

Consider a dynamic control system described by a differential equation with impulsive control (measure) $u(\cdot)$:

$$dx = A(t)xdt + du, \qquad x(-0) = x_0, \qquad (1)$$

or in the integral form [14],

$$x(t) = x(t; u(\cdot), x_0) = X(t)x_0 + + \int_0^t X(t)X^{-1}(\tau)du(\tau). \qquad (2)$$

Here we assume that $A(t)$ is continuous $n \times n$ - matrix function, $X(t)$ is the fundamental matrix solution $\dot{X} = A(t)X$ $(X(0) = I)$, $u(\cdot) \in V_p^n$ $(1 \leq p < \infty)$ where $V_p^n$ means the space of $n$-vector functions $u(\cdot)$ such that $u(t)$ is continuous from the right on $[0, T)$ with $u(-0) = 0$ and

$$V_p[u(\cdot)] = \sup_{\{t_i | 0 = t_0 < \ldots < t_k = T\}} \sum_{i=1}^{k} \|u(t_i) - u(t_{i-1})\|_p < \infty$$

$$\|u\|_p = \left( \sum_{i=1}^{n} |u_i|^p \right)^{\frac{1}{p}}, \quad u = (u_1, \ldots, u_n).$$

Let $\mathcal{E}_0$ be an ellipsoid in $R^n$:

$$\mathcal{E}_0 = \{ l \in R^n \mid l' Q_0 l \leq 1 \}, \qquad (3)$$

where $Q_0$ is a given symmetric positive definite $n \times n$ matrix.

Denote $C_q^n$ the space of continuous $n$-vector functions $y(\cdot)$ with the norm

$$\|y(\cdot)\|_{\infty,q} = \max_{0 \leq t \leq T} \|y(t)\|_q.$$

It is well known that the space $V_p^n = C_q^{n*}$ where $p = 1$ if $q = \infty$, $p = \infty$ if $q = 1$ and $1 < p < \infty$ if $q = (1 - p^{-1})^{-1}$.

Consider the so-called "ellipsoid" $\mathcal{E}$ in $C_q^n$:

$$\mathcal{E} = \{ y(\cdot) \in C_q^n \mid y'(t) Q_0 y(t) \leq 1 \;\; \forall t \in [0, T] \} = = \{ y(\cdot) \in C_q^n \mid y(t) \in \mathcal{E}_0 \;\; \forall t \in [0, T] \} \qquad (4)$$

and its conjugate $\mathcal{E}^* \subset V_p^m$,

$$\mathcal{E}^* = \{ u(\cdot) \in V_p^m \| \int_0^T y(t) du(t) \leq 1 \qquad (5)$$

$$\forall y(t) \in \mathcal{E}, t \in [0, T] \}.$$

**Definition 1.** The function $u(\cdot) \in V_p^m$ will be called the admissible control if $u(\cdot) \in U = \mathcal{E}^*$.

Let $u(\cdot)$ be a piecewise constant function on $[0, T]$ with discontinuity instants $\{t_i\}$ and with

$$\Delta u = u(t_{i+1}) - u(t_i) \in \mathcal{E}_0^* = \{ z \in R^n | \; z' Q_0^{-1} z \leq 1 \}. \qquad (6)$$

Then $u(\cdot)$ is admissible.

We will assume also that the initial value $x_0$ to the system (1) is unknown but bounded with a given bound $x_0 \in \mathcal{X}_0$,

$$\mathcal{X}_0 = \{ x_0 \mid x_0' R^{-1} x_0 \leq 1 \} \qquad (7)$$

where $R$ is a symmetric positive definite $n \times n$ matrix.

Denote

$$\mathcal{X}(t; \mathcal{X}_0) = \bigcup_{x_0 \in \mathcal{X}_0} \bigcup_{u \in U} x(t; u(\cdot), x_0).$$

**Definition 2.** The set $\mathcal{X}(t; \mathcal{X}_0)$ is called the reachable set of the impulsive differential system (1) from the initial set $\mathcal{X}_0$ at instant $t$.

So the main problem of the paper is to find the external estimates of ellipsoidal type for the reachable set $\mathcal{X}(T; \mathcal{X}_0)$ basing on the special ellipsoidal structure of sets $\mathcal{X}_0$ and $U$.

## 3 Preliminaries

We mention here some results concerning the properties of $\mathcal{X}(T; \mathcal{X}_0)$. It is known [12, 13] that $\mathcal{X}(T; \mathcal{X}_0)$ is convex and compact in $R^n$ and the following theorems are valid.

**Theorem 1.** Let $l_0 \neq 0$, $u^0 \in U$ $x_0 \in \mathcal{X}_0$ be such that

$$\rho(l_0 | \; \mathcal{X}(T; \mathcal{X}_0)) = \max_{x \in \mathcal{X}(T, \mathcal{X}_0)} l_0' x = = l_0' x(T; u^0, x_0); \quad X(T, \tau) = X(T) X^{-1}(\tau).$$

Then

$$\int_0^T l_0' X(T, \tau) du^0(\tau) = \max_{u \in U} \int_0^T l_0' X(T, \tau) du(\tau)$$

and

$$l_0' X(T, 0) x_0 = \max_{x \in \mathcal{X}_0} l_0' X(T, 0) x.$$

**Theorem 2.** For any point $x$ that belongs to the boundary of the reachable set $\mathcal{X}(T; \mathcal{X}_0)$ there exists an admissible piecewise constant control function $\tilde{u}(\cdot) \in U$ with no more than $n$ points $\{t_i\}$ of discontinuity such that (1) $x(T; \tilde{u}, x_0) = x$ with some $x_0 \in \mathcal{X}_0$ and (2) the jumps $\Delta_i \tilde{u} = \tilde{u}(t_{i+1}) - \tilde{u}(t_i)$ of $\tilde{u}(\cdot)$ belong to the ellipsoid $\mathcal{E}^*$.

**Example 1.** Consider the following control system:

$$\begin{cases} dx_1(t) &= x_2(t)dt + du_1(t), \\ dx_2(t) &= du_2(t), \end{cases} \quad (8)$$

Here $\mathcal{X}_0 = \{0\}$ and the set $U$ is generated by the ellipsoid

$$\mathcal{E}_0 = \left\{ l \in R^2 \mid \quad l' Q_0 l \leq 1 \right\},$$

$$Q_0 = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}, \quad a, b > 0.$$

From Theorems 1-2 we have:

$$\rho(l | \ \mathcal{X}(T; \mathcal{X}_0)) = \int_0^T l' X(T, \tau) du^0(\tau) =$$

$$= \max_{0 \leq \tau \leq T} (G(\tau, l))^{\frac{1}{2}} \quad (9)$$

where $G(\tau, l) = l' X(T, \tau) Q_0 X'(T, \tau) l$. The reachable set $\mathcal{X}(T; 0)$ is given at the Fig. 1 where we denote

$$x_1^* = \frac{a^2}{\sqrt{a^2 + 0.25 T^2 b^2}} < a,$$

$$x_2^* = \frac{a^2 + 0.5 T^2 b^2}{\sqrt{a^2 + 0.25 T^2 b^2}} > Tb,$$

$$\mathcal{E}_1 = \{ x \in R^2 \mid \frac{1}{a^2}(x_1 - x_2 T)^2 + \frac{x_2^2}{b^2} \leq 1 \},$$

$$\mathcal{E}_2 = \{ x | \ \ x' Q_0^{-1} x \leq 1 \}.$$

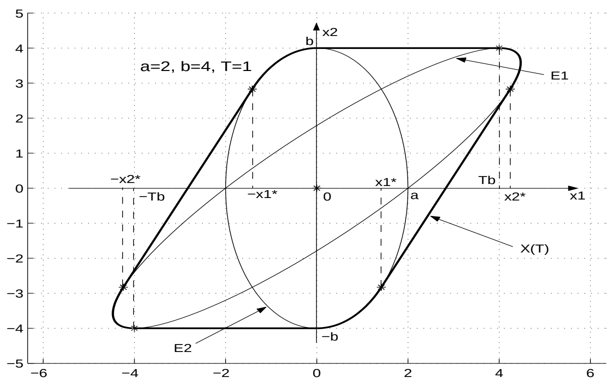

Figure 1. The reachable set $\mathcal{X}(T) = \mathcal{X}(T; \mathcal{X}_0)$.

# 4 Main Results

In this section we apply the techniques of the ellipsoidal calculus to find the external estimates for $\mathcal{X}(T; \ \mathcal{X}_0)$.

## 4.1 First Approach

We take $\mathcal{X}_0 = \{0\}$ first and denote $\mathcal{X}(T) = \mathcal{X}(T; \{0\})$ and

$$T_* = \{ \tau_* \in [0, T] \mid \exists l_* \neq 0, (G(\tau_*, l_*))^{\frac{1}{2}} = \max_{0 \leq \tau \leq T} (G(\tau, l_*))^{\frac{1}{2}} \} \quad (10)$$

with $G(\tau, l)$ defined in the above section.
**Assumption P.** We will assume further that the set $T_*$ is finite:

$$T_* = \{ \tau_{*1}, \tau_{*2}, \ldots, \tau_{*m} \} \subset [0, T].$$

The class of systems for which this assumption is valid is not empty, e.g. it is fulfilled in Example 1.

From Theorem 2 we have

**Theorem 3.** Under the assumption P we have

$$\mathcal{X}(T) = co \bigcup_{\tau \in T_*} \mathcal{E}(0, Q_\tau), \quad (11)$$

$$\mathcal{E}(0, Q_\tau) = \{ x \in R^n \mid x' Q_\tau^{-1} x \leq 1 \},$$
$$Q_\tau = X(T, \tau) Q_0 X'(T, \tau).$$

Applying the results [3, 4] we have from Theorem 3
**Theorem 4.** For each $p = (p_1, \ldots, p_m) > 0$

$$\mathcal{X}(T) \subseteq \mathcal{E}(0, Q_p^+) \bigcap S(0, K) \quad (12)$$

where

$$Q_p^+ = (p_1 + p_2 + \ldots + p_m)(p_1^{-1} Q_{\tau_1} +$$
$$+ p_2^{-1} Q_{\tau_2} + \ldots + p_m^{-1} Q_{\tau_m}), \quad (13)$$

$$S(0,K) = \{x \in R^n | ||x|| \le K\},$$

$$K = \max_{l'l=1, \tau \in T_*} (l'Q_\tau l)^{\frac{1}{2}}. \quad (14)$$

This first approach is illustrated by the following example.

**Example 2.** Consider again the system (8) of the example 1. Let now $a = b = T = 1$. In this case we can easily find the ball $S(0;K)$ and the constant $K = \sqrt{\frac{3+\sqrt{5}}{2}}$. The upper ellipsoid $\mathcal{E}(0, Q_p^+)$ for the sum $\mathcal{E}_0 + \mathcal{E}_1$ is defined by

$$Q_p^+ = (1 + p^{-1})Q_0 + (1 + p)Q_1,$$

$$Q_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

$$\sqrt{\lambda_{min}} \le p \le \sqrt{\lambda_{max}},$$

where $\lambda_{min}(\lambda_{max})$ is a minimal (maximal) root of equation $|Q_0 - \lambda Q_1| = 0$. Therefore $\lambda_{min} = \sqrt{\frac{3-\sqrt{5}}{2}}$ and $\lambda_{max} = \sqrt{\frac{3+\sqrt{5}}{2}}$. Then $\mathcal{X}(1,0) \subset S(0,K) \cap \mathcal{E}_p(0, Q_p^+)$ (see Fig.2 with $p = \sqrt{2}$).

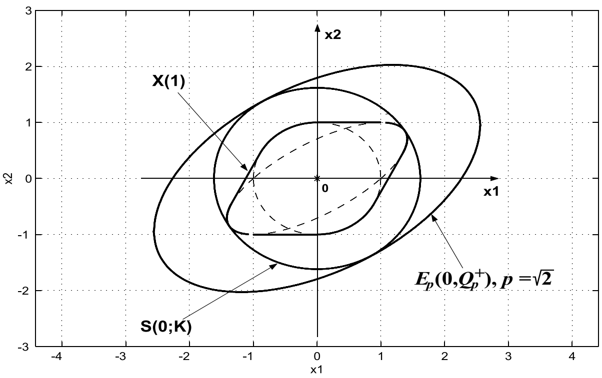


Figure 2. First estimate of $\mathcal{X}(T)$.

## 4.2 Second Approach

Consider first the auxiliary problem.

**Problem AP.** Given two ellipsoids

$$\mathcal{E}_0 = \{x \in R^n \mid x'Q_0^{-1}x \le 1\},$$

$$\mathcal{E}_1 = \{x \in R^n \mid x'Q_1^{-1}x \le 1\}, \quad (15)$$

find an ellipsoid

$$\mathcal{E}_2 = \{x \in R^n \mid x'Q_2^{-1}x \le 1\}$$

that contains $\mathcal{E}_0 \cup \mathcal{E}_1$ (therefore, the $\mathcal{E}_2$ will contain the convex hull $co(\mathcal{E}_0 \cup \mathcal{E}_1)$).

Equivalently, we need to find a symmetric positive definite matrix $Q_2$ such that for all $l \in R^n$

$$(l'Q_0 l)^{\frac{1}{2}} \le (l'Q_2 l)^{\frac{1}{2}}, \quad (l'Q_1 l)^{\frac{1}{2}} \le (l'Q_2 l)^{\frac{1}{2}}, \quad (16)$$

and it is desirable also that the ellipsoid $\mathcal{E}_2$ was of minimal possible volume [4].

We need to do three consequent steps to solve the Problem AP.

*Step 1.* Let $\lambda_1 \dots \lambda_n$ be the roots of the equation

$$|Q_0\lambda - Q_1| = 0 \quad (17)$$

(note that $\lambda_i > 0$ $(i = 1 \dots n)$ are the eigenvalues of matrix $Q_0^{-1}Q_1$).

Denote $B = Q_0^{-\frac{1}{2}} Q_1 Q_0^{-\frac{1}{2}}$. The matrix $B$ is also symmetric and positive definite [16] and there exists an orthogonal matrix $T$ such that [16]

$$T'BT = diag\{\lambda_1, \dots, \lambda_n\} = W^2, \quad (18)$$

$$(TT' = T'T = I).$$

We transform the coordinates from the vector $x$ to the new variable $s$ that satisfies the equality $x = Q_0^{\frac{1}{2}} Ts$. Under this transformation the ellipsoids $\mathcal{E}_0$, $\mathcal{E}_1$ (15) become the ellipsoids

$$\tilde{\mathcal{E}}_0 = \{s \in R^n \mid s's \le 1\}, \quad (19)$$

$$\tilde{\mathcal{E}}_1 = \{s \in R^n \mid s'(W^2)^{-1}s \le 1\}$$

where $W^2$ determined in (18).

*Step 2.* We construct the ellipsoid $\tilde{\mathcal{E}}_2 \supseteq \tilde{\mathcal{E}}_0 \cup \tilde{\mathcal{E}}_1$ where

$$\tilde{\mathcal{E}}_2 = \{s \in R^n \mid s'\tilde{Q}_2^{-1}s \le 1\},$$

$$\tilde{Q}_2 = diag\{\mu_1^2, \dots, \mu_n^2\}$$

$$\mu_i^2 = \max\{1, \lambda_i\}, \quad i = 1, 2, \dots, n.$$

**Theorem 5.** Ellipsoid $\tilde{\mathcal{E}}_2$ is minimal with respect to inclusion among all ellipsoids containing $\tilde{\mathcal{E}}_0 \bigcup \tilde{\mathcal{E}}_1$.

The proof of this theorem follows from the properties of ellipsoids and the result ([4], corollary 5.1).

*Step 3.* We return to the space of $x$ - coordinates and therefore we get the inclusion

$$\mathcal{E}_2 = \{x \in R^n \mid x'Q_2^{-1}x \le 1\} \supseteq (\mathcal{E}_0 \cup \mathcal{E}_1),$$

$$Q_2 = Q_0^{\frac{1}{2}} T \tilde{Q}_2 T' Q_0^{\frac{1}{2}}.$$

It should be noted that applying this 3-steps procedure consequently we can solve the main estimation problem under the above assumption P that guarantees the finite number of such steps. The second approach allows to estimate the reachable set more precisely than the first one, but it takes more calculations. The combination of methods 1 and 2 allows to find more exact estimates.

**Example 3.** Let us apply the above scheme to the system (8) of Examples 1-2. Here we take

$$Q_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

According to steps 1-3 we need to find first an orthogonal matrix $T$. We have in this case

$$T = \begin{pmatrix} \sqrt{\frac{1+\sqrt{5}}{2\sqrt{5}}} & -\sqrt{\frac{2}{\sqrt{5}(1+\sqrt{5})}} \\ \sqrt{\frac{2}{\sqrt{5}(1+\sqrt{5})}} & \sqrt{\frac{1+\sqrt{5}}{2\sqrt{5}}} \end{pmatrix},$$

$$\mathcal{E}_0 = \{x \mid x'Q_0^{-1}x \le 1\} = T\tilde{\mathcal{E}}_0, \ \tilde{\mathcal{E}}_0 = \{s \mid s's \le 1\},$$

$$\mathcal{E}_1 = \{x \mid x'Q_1^{-1}x \le 1\} = T\tilde{\mathcal{E}}_1,$$

$$\tilde{\mathcal{E}}_1 = \{s \mid s'(W^2)^{-1}s \le 1\},$$

$$\tilde{\mathcal{E}}_2 = \{s \mid s'\tilde{Q}_2^{-1}s \le 1\} \supset \tilde{\mathcal{E}}_0 \cup \tilde{\mathcal{E}}_1,$$

$$\tilde{Q}_2 = \begin{pmatrix} \sqrt{\frac{3+\sqrt{5}}{2}} & 0 \\ 0 & 1 \end{pmatrix}.$$

The Fig.3 illustrates the above inclusion.

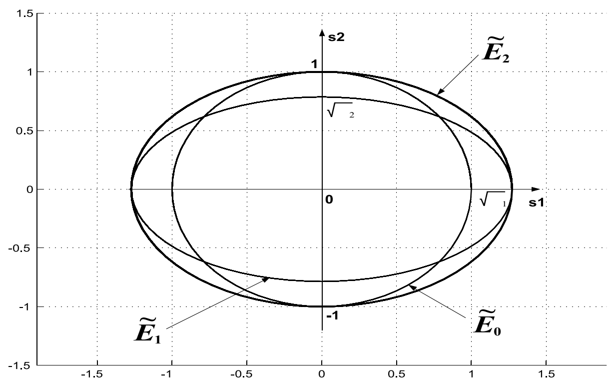


Figure 3. Auxiliary ellipsoids $\tilde{\mathcal{E}}_0, \tilde{\mathcal{E}}_1, \tilde{\mathcal{E}}_2$.

Therefore we get the inclusion

$$\mathcal{E}_2 = \{x \mid x'Q_2^{-1}x \le 1\} \supset \mathcal{E}_0 \cup \mathcal{E}_1, \quad Q_2 = TQ_2T'$$

and finally we have

$$Q_2 = \frac{1+\sqrt{5}}{\sqrt{5}} \begin{pmatrix} 1.5 & 0.5 \\ 0.5 & 1 \end{pmatrix}.$$

The inclusion $\mathcal{E}_2 \supset \mathcal{E}_0 \cup \mathcal{E}_1$ is shown at Fig. 4. Comparing Fig. 4 with Fig.1-2 we get $\mathcal{X}(1) \subset \mathcal{E}_2$.

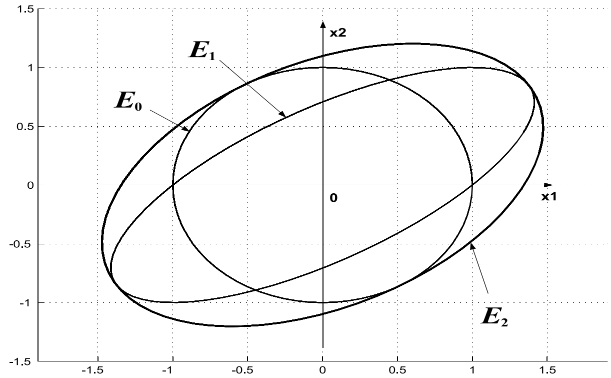


Figure 4. The second estimate of $\mathcal{X}(T)$.

## 5 Conclusion

We presented here some approaches that allow to find the external set-valued estimates of the reachable sets of linear impulsive control systems. The examples that illustrate the techniques of ellipsoidal calculus discussed in the paper are also given.

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