Discontinuous Solutions of Differential Equations With Time Delay*

Yu. V. FETISOVA , A. N. SESEKIN,
Department of Optimal Control
Institute of Mathematics and Mechanics of the Russian Academy of Sciences
16 S.Kovalevskaya Str., Ekaterinburg 620219
RUSSIA
http://www.imm.uran.ru

Abstract: - The paper deals with the introduction of notion of discontinuous solution for nonlinear system with distributions in the right hand side of the system and with time delay. The definition of solution based on the closure of the set of smooth solutions in the space of functions of bounded variation. Sufficient conditions of existence and continuous dependence on initial function so defined solutions are received. The specified conditions for existence of solutions for linear systems with distributions in a matrix of system is obtained. For such systems it is received the representation of solution of Cauchy problem.

Key-Words: - Differential equations, distributions, impulsive functions, functions of bounded variation, Cauchy problem, system with time delay.

1 Introduction

This paper devote to the problem of definition of solution for nonlinear system with distributions in right hand side of the system and with time delay. The question on definition of the solution in such systems without time delay was considered in [1, 2, 3, 4, 5, 6, 7, 8, 9]. This works use approach based on the definition of solution as a limit of a sequence of smooth solutions, it was generated by smooth approximations of distributions in the right side of the system. This approach is natural from the point of view of the theory of control [10] where impulsive control frequently are the idealized processes with the large changes of parameters for the short time intervals.

This approach is realized here for systems with time delay. Also, the linear system with the generalized functions in a matrix of the system and with time delay is considered. Sufficient conditions of existence so defined solutions are received. The theorem about continuous dependence on initial function of discontinuous solutions is proved. The formula for representation of solution for the Cauchy problem is established.

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2 Discontinuous solutions of nonlinear differential equations with time delay

We consider the following system of the differential equations

$$\dot{x} = f(t, x(t), x(t-\tau)) + B(t, x(t))\dot{v}(t), \qquad (1)$$

here $x(t)$, $v(t)$ are accordingly n and m - dimensional vector functions of time, $f(t,x,y)$ is an n - dimensional vector-valued function, $B(t,x)$ is $n\times m$ matrix-valued function, $\tau > 0$ is a constant delay, $\varphi(t)$ is an initial function (initial condition) of bounded variation given on $[t_0-\tau,\ t_0]$. Further we assume that $f(t,x,y)$ is continuous in t and Lipschitz in x, y, $B(t,x)$ continuous in t and Lipschitz in x with constant L.

If $v(t)$ is absolutely continuous and if the inequalities

$$\|f(t,x,y)\| \le \kappa(1+\|x\|),\ \ \|B(t,x)\| \le \kappa(1+\|x\|)$$

holds for all admissible t, x and y, then a solution of equation (1) on $[t_0, \vartheta]$ can be constructed by means of the steps method [11].

If $v(t)$ is a function of bounded variation, then the derivatives in (1) are necessary to be considered in a generalized sense. If the function $v(t)$ is discontinuous at t^*, then the system will be acted by an impulse at the moment t^*. Thus, we meet the problem of multiplication of discontinuous functions by generalized ones in the term $B(t,x(t))\dot{v}(t)$.

Definition 1. A vector-valued function of bounded variation $x(t)$ will be called an approximable solution to (1), if $x(t)$ is a pointwise limit on $[t_0,\, \vartheta]$ of the sequence $x_k(t)$, where $x_k(t)$ are the absolutely continuous solutions of equation (1) with absolutely continuous functions $v_k(t)$, pointwise converging to $v(t)$ if $x(t)$ does not depend on the choice of sequence $v_k(t)$.

This definition differs from definition of the discontinuous solution entered in [9] for the of neutral type differential equation because in this case there is no necessity to approximate initial function.

Theorem 1. Let $f(t,x,y)$ and $B(t,x)$ satisfy the conditions mentioned above. In addition to it, we assume that there exist the partial derivatives $\partial b_{i,j}/\partial x_\nu$ that is continuous on x, which satisfy the following equalities

$$\sum_{\nu=1}^{n} \frac{\partial b_{ij}(t,x)}{\partial x_\nu} b_{\nu l}(t,x) = \sum_{\nu=1}^{n} \frac{\partial b_{il}(t,x)}{\partial x_\nu} b_{\nu j}(t,x)$$

(Frobenius condition) $i = 1,2,...,n; j,l = 1,2,...,m.$ Then for any vector function of bounded variation $v(t)$ there exists the approximable solution $x(t)$ of (1) such that this solution satisfies to the integral equation

$$x(t) = \varphi(t_0) + \int_{t_0}^{t} f(\xi, x(\xi), x(\xi-\tau))\, d\xi$$

$$+ \int_{t_0}^{t} B(\xi, x(\xi))\, dv^c(\xi)$$

$$+ \sum_{t_i \le t,\, t_i \in W_-} S(t_i, x(t_i - 0), \Delta v(t_i - 0))$$

$$+ \sum_{t_i < t,\, t_i \in W_+} S(t_i, x(t_i), \Delta v(t_i + 0)). \qquad (2)$$

Here $v^c(t)$ is an continuous part of the function of bounded variation $v(t)$,

$$S(t,x,\Delta v) = z(1) - z(0),$$

$$\dot z(\xi) = B(t, z(\xi))\Delta v(t), \qquad z(0) = x,$$

and the set W_- (W_+) is the set of points in which the function $v(t)$ is discontinuous from the left (from the right),

$$\Delta v(t-0) = v(t) - v(t-0), \;\; \Delta v(t+0) = v(t+0) - v(t).$$

Proof. On an interval $t_0 \le t \le t_0 + \tau$ the system (1) takes the form

$$\dot x = f(t, x(t), \varphi(t-\tau)) + B(t, x(t))\dot v(t).$$

$\varphi(t)$ is the known function. Then for this system we can apply the result [9]. According to [9] under the assumptions of this theorem it is possible to conclude, that system (1) on the interval $[t_0, t_0+\tau]$ has the solution. Furthermore, this solution will satisfy to equation (2), where we must change $x(\xi-\tau)$ on $\varphi(\xi-\tau)$. Then we can construct the solution on $[t_0, \vartheta]$ by the steps method [11].

3 Continuous dependence on initial function of discontinuous solutions for differential equations with time delay

Now we shall give the following

Definition 2. Suppose $\varphi_k(t)$ is a sequence of initial functions pointwisely converging to a function $\varphi(t)$, $x_k(t)$ is a sequence of approximable solutions of system (1) generated by the sequence $\varphi_k(t)$, and $x(t)$ is an approximable solution of the same system corresponding to the initial function $\varphi(t)$. We shall say, that the solution $x(t)$ depends continuously on the initial function φ, if $\forall\ t \in [t_0, t_f]\ \ x_k(t)$ pointwisely converges to $x(t)$ as $k \to \infty$.

Theorem 2. Let in a region of variables $t \in [t_0, t_f]$, x and $y \in R^n$ a vector function $f(t,x,y)$ and matrix function $B(t,x)$ satisfy the conditions of theorem 1. In addition to it, the following inequality holds

$$|f(t,x,y) - f(t,x_*,y_*)| \le L|x - x_*| + M|y - y_*|.$$

Then the approximable solution $x(t)$ of system (1) depends continuously on the initial function $\varphi(t)$.

Proof. Let $x_\varphi(t)$ and $x^*_\varphi(t)$ be approximable solutions of system (1) corresponding to any initial functions of bounded variation φ and φ^*. Let's denote a difference $\varphi(t_0) - \varphi^*(t_0)$ by $\Delta\varphi(t_0)$, then we can write

$$x_\varphi(t) - x_{\varphi^*}(t) = \Delta\varphi(t_0) + \int_{t_0}^{t} (f(\xi, x_\varphi(\xi), x_\varphi(\xi-\tau))$$

$$- f(\xi, x_{\varphi^*}(\xi), x_{\varphi^*}(\xi-\tau)))\, d\xi$$

$$+ \int_{t_0}^{t} (B(\xi, x_\varphi(\xi)) - B(\xi, x_{\varphi^*}(\xi)))\, dv^c(\xi)$$

$$+\sum_{t_i\le t, t_i\in W_-}(S(t_i,x_\varphi(t_i-0),\Delta v(t_i-0))$$

$$-S(t_i,x_{\varphi^*}(t_i-0),\Delta v(t_i-0)))$$

$$+\sum_{t_i<t, t_i\in W_+}(S(t_i,x_\varphi(t_i),\Delta v(t_i)$$

$$-S(t_i,x_{\varphi^*}(t_i),\Delta v(t_i))). \qquad (3)$$

According to [9] the following estimation is valid

$$|S(t_i,x_\varphi(t_i-0),\Delta v(t_i-0))$$

$$-S(t_i,x_\varphi^*(t_i-0),\Delta v(t_i-0))|\le$$

$$\le (e^{K|\Delta v(t_i-0)|}-1)\,|x_\varphi(t_i-0)-x_\varphi^*(t_i-0)|.$$

Then it is possible to estimate the sum

$$\sum_{t_i\le t, t_i\in W_-}|S(t_i,x_\varphi(t_i-0),\Delta v(t_i-0))$$

$$-S(t_i,x_{\varphi^*}(t_i-0),\Delta v(t_i-0))|\le$$

$$\le \sum_{t_i\le t, t_i\in W_-}(e^{K|\Delta v(t_i-0)|}-1)*$$

$$*|x_\varphi(t_i-0)-x_{\varphi^*}(t_i-0)|. \qquad (4)$$

The similar estimation is valid and for 'right jumps' in points t_i.

If we compute modules of the left and the right sides of equality (3) and estimate the right module by using 'triangle inequalities', Lipschitz condition for functions $f(t,x,y)$ and $B(t,x)$ and estimation (4), then we shall have

$$|x_\varphi(t)-x_{\varphi^*}(t)|\le|\Delta\varphi(t_0)|+L\int_{t_0}^{t}|x_\varphi(\xi)$$

$$-x_{\varphi^*}(\xi)|\,d\xi+M\int_{t_0}^{t}|x_\varphi(\xi-\tau)-x_{\varphi^*}(\xi-\tau)|\,d\xi$$

$$+L\int_{t_0}^{t}|x_\varphi(\xi)-x_{\varphi^*}(\xi)|\,|dv^c(\xi)|$$

$$+\sum_{t_i<t, t_i\in W_+}(e^{K|\Delta v(t_i)|}-1)\,|x_\varphi(t_i)-x_{\varphi^*}(t_i,)|$$

$$+\sum_{t_i\le t, t_i\in W_-}(e^{K|\Delta v(t_i-0)|}-1)\,|x_\varphi(t_i-0)$$

$$-x_{\varphi^*}(t_i-0)|\le|\Delta\varphi(t_0)|+L\int_{t_0}^{t}|x_\varphi(\xi)-x_{\varphi^*}(\xi)|\,d(\xi$$

$$+\operatorname*{var}_{[t_0,\,\xi]}v^c(\cdot))+M\int_{t_0}^{t}|x_\varphi(\xi-\tau)-x_{\varphi^*}(\xi-\tau)|\,d\xi$$

$$+\sum_{t_i<t, t_i\in W_+}(e^{L|\Delta v(t_i)|}-1)|x_\varphi(t_i)-x_{\varphi^*}(t_i)|$$

$$+\sum_{t_i\le t, t_i\in W_-}(e^{L|\Delta v(t_i-0)|}-1)*$$

$$*|x_\varphi(t_i-0)-x_{\varphi^*}(t_i-0)|. \qquad (5)$$

For $t\in[t_0,t_0+\tau]$ inequality (5) takes the form

$$|x_\varphi(t)-x_{\varphi^*}(t,)|\le|\Delta\varphi(t_0)|+L\int_{t_0}^{t}|x_\varphi(\xi)$$

$$-x_{\varphi^*}(\xi)|\,d(\xi+\operatorname*{var}_{[t_0,\,\xi]}v^c(\cdot))$$

$$+M\int_{t_0}^{t}|\varphi(\xi-\tau)-\varphi^*(\xi-\tau)|\,d\xi$$

$$+\sum_{t_i<t, t_i\in W_+}(e^{L|\Delta v(t_i)|}-1)|x_\varphi(t_i)-x_{\varphi^*}(t_i)|$$

$$+\sum_{t_i\le t, t_i\in W_-}(e^{L|\Delta v(t_i-0)|}-1)*$$

$$|x_\varphi(t_i-0)-x_{\varphi^*}(t_i-0)|. \qquad (6)$$

According to [9] the solution of integral inequality (6) can be estimated like

$$|x_\varphi(t)-x_{\varphi^*}(t)|\le e^{L\cdot P(t)}[|\Delta\varphi(t_0)|$$

$$+M\int_{t_0}^{t}e^{-L\cdot P(s)}|\varphi(s-\tau)-\varphi^*(s-\tau)|\,ds].$$

Here

$$P(t)=t-t_0+\operatorname*{var}_{[t_0,\,t]}v^c+\sum_{t_i<t, t_i\in W_+}|\Delta v(t_i)|$$

$$+\sum_{t_i\le t, t_i\in W_-}|\Delta v(t_i-0)|.$$

It is clear that for any $t\in[t_0,t_0+\tau]$ $x_{\varphi^*}(t)$ pointwisely converges to $x_\varphi(t)$ as $\varphi^*(t)$ pointwisely converges to $\varphi(t)$.

For the following step when $t\in[t_0+\tau,t_0+2\tau]$ we have

$$|x_\varphi(t)-x_{\varphi^*}(t)|\le|x_\varphi(t_0+\tau)-x_{\varphi^*}(t_0+\tau)|$$

$$+L\int_{t_0+\tau}^{t}|x_\varphi(\xi)-x_{\varphi^*}(\xi)|\,d(\xi+\operatorname*{var}_{[t_0+\tau,\,\xi]}v^c(\cdot))$$

$$+M\int\limits_{t_0+\tau}^{t}|x_\varphi(\xi-\tau)-x_{\varphi^*}(\xi-\tau)|\,d\xi]$$

$$+\sum_{t_i\le t, t_i\in W_-}(e^{L|\Delta v(t_i-0)|}-1)|x_\varphi(t_i-0)-x_{\varphi^*}(t_i-0)|$$

$$+\sum_{t_i<t, t_i\in W_+}(e^{L|\Delta v(t_i)|}-1)|x_\varphi(t_i)-x_{\varphi^*}(t_i)|. \quad (7)$$

Applying the estimation from [9] mentioned above to the solution of inequality (7) we can write

$$|x_\varphi(t)-x_{\varphi^*}(t)|\le e^{L\cdot P(t)}[|\Delta\varphi(t_0)|$$

$$+M\int\limits_{t_0}^{t}e^{-L\cdot P(s)}|x_\varphi(s-\tau)-x_{\varphi^*}(s-\tau)|\,ds].$$

From the results of the first step it follows that for any $t\in[t_0+\tau,t_0+2\tau]$ $x_{\varphi^*}(t)$ pointwisely converges to $x_\varphi(t)$ as $\varphi^*(t)$ pointwisely converges to $\varphi(t)$. Continuing in the same way, we see that this statement is valid for any $t\in[t_0,t_f]$. So it is possible to conclude, that the approximable solution $x(t)$ for system of differential equations (1) depends continuously on the initial function $\varphi(t)$.

4 Cauchy formula for linear differential equations with distributions in a matrix of system and with time delay

In this section we consider a linear system of the differential equations

$$\dot{x}(t)=\bar{A}(t)x(t)+C(t)x(t-\tau)+h(t), \quad (8)$$

where $\bar{A}(t)=A(t)+\sum\limits_{i=1}^{m}D_i(t)\dot{v}_i(t)$, $A(t)$ and $C(t)$ are continuous $n\times n$-matrix functions, $h(t)$ is a vector function with summable elements, $D_i(t)$ are continuous $n\times n$ - matrix functions for any $i\in\overline{1,m}$, $v_i(t)$ are components of a vector function of bounded variation $v(t)=(v_1(t),v_2(t)...,v_m(t))^T$, $\tau>0$ is a constant time delay, $\varphi(t)$ is an initial function - the function of bounded variation defined on $[t_0-\tau,\,t_0]$.

Applying theorem 1 to system (8) we shall obtain

Theorem 3. Let the matrixes $D_i(t)\,(i\in\overline{1,m})$ be mutually commutative for all $t\in[t_0,\vartheta]$. Then there exists an approximable solution $x(t)$ of (8), which satisfies to the integral equation

$$x(t)=\varphi(t_0)+\int\limits_{t_0}^{t}A(\xi)x(\xi)\,d\xi+\int\limits_{t_0}^{t}h(\xi)\,d\xi$$

$$+\sum_{i=1}^{m}\int\limits_{t_0}^{t}D_i(\xi)x(\xi)\,dv_i^c(\xi)+\int\limits_{t_0}^{t}C(\xi)x(\xi-\tau)\,d\xi$$

$$+\sum_{t_i\le t,\,t_i\in W_-}\tilde{S}(t_i,x(t_i-0),\Delta v(t_i-0))$$

$$+\sum_{t_i<t,\,t_i\in W_+}\tilde{S}(t_i,x(t_i),\Delta v(t_i+0)).$$

Here

$$\tilde{S}(t,x,\Delta v)=z(1)-z(0),$$

$$\dot{z}(\xi)=\sum_{i=1}^{m}D_i(t)x(\xi)\,\Delta v_i(t), \qquad z(0)=x.$$

Theorem 4. By assumptions of theorem 3 the approximable solution of system (8) represents as

$$x(t)=x^{hom}(t)+\int\limits_{t_0}^{t}U(t,s)h(s)\,ds,$$

where $x^{hom}(t)$ is the solution of corresponding homogeneous system, $U(t,s)$ is a fundamental matrix of system (8) being the solution of the integral equation

$$U(t,s)=\begin{cases}E+\int\limits_{s}^{t}A(\xi)U(\xi,s)\,d\xi\\ +\sum\limits_{i=1}^{m}\int\limits_{s}^{t}D_i(\xi)U(\xi,s)\,dv_i^c(\xi)\\ +\int\limits_{s}^{t}C(\xi)U(\xi-\tau,s)d\xi\\ +\sum\limits_{t_i\le t,\,t_i\in W_-}\tilde{S}(t_i,U(t_i-0,s),\\ \Delta v(t_i-0))+\sum\limits_{t_i<t,\,t_i\in W_+}\tilde{S}(t_i,U(t_i,s),\\ \Delta v(t_i)),\ t\ge s,\\ 0,\ s-\tau\le t<s.\end{cases} \quad (9)$$

Here the functions of jumps $\tilde{S}(t,U(t,s),\Delta v(t))$ are defined by means of equations

$$\tilde{S}(t,U(t,s),\Delta v(t))=z(1)-z(0),$$

$$\dot{z}(\xi)=\sum_{i=1}^{m}D_i(t)z(\xi)\Delta v_i(t), \quad z(0)=U(t,s) \quad (10)$$

Proof. Let $v_k(t)$ will be a sequence of absolutely continuous functions, pointwisely converging to a function of bounded variation $v(t)$. If system (8) is written down for any element of the sequence $v_k(t)$, then we can apply the theorem of variations on constants from [12]. Any element of the sequence of absolutely continuous solutions $x_k(t)$ of system (8) satisfy the following equation

$$x_k(t) = x_k^{hom}(t) + \int\limits_{t_0}^{t} U_k(t,s)h(s)\,ds, \qquad (11)$$

here $x_k^{hom}(t)$ is the solution of the corresponding homogeneous system of the differential equations and $U_k(t,s)$ is the solution of the integral equation

$$U_k(t,s) = \begin{cases} E + \int\limits_s^t A(\xi)U_k(\xi,s)\,d\xi \\ + \sum\limits_{i=1}^{m} \int\limits_s^t D_i(\xi)U_k(\xi,s)\,dv_{i_k}(\xi) \\ + \int\limits_s^t C(\xi)U_k(\xi-\tau,s)\,d\xi,\ t \ge s, \\ 0,\ s-\tau \le t < s. \end{cases}$$

It is not difficult to show, that the sequence $x_k(t)$ is bounded and the sequence of variations of vector functions $x_k(t)$ on $[t_0, \vartheta]$ is bounded also. Let the sequence $x_k(t)$ on $[t_0, \vartheta]$ pointwisely converges to $x(t)$. Otherwise, according to Helly theorem [13], it is possible to select a converging subsequence from the sequence $x_k(t)$. Similarly, the sequence $U_k(t,s)$ pointwisely converges to certain matrix function $U(t,s)$ such that $U(t,s)$ is the function of bounded variation both on a variable t, and on a variable s.

Then, according to Lebesgue theorem [13], in equality (11) it is possible to pass to the limits as $k \longrightarrow \infty$. Then we obtain

$$x(t) = \lim_{k\to\infty} x_k^{hom}(t) + \lim_{k\to\infty} \int\limits_{t_0}^{t} U_k(t,s)h(s)\,ds$$

$$= x^{hom}(t) + \int\limits_{t_0}^{t} \lim_{k\to\infty} U_k(t,s)h(s)\,ds =$$

$$= x^{hom}(t) + \int\limits_{t_0}^{t} U(t,s)h(s)\,ds,$$

where $U_k(t,s)$ is the solution of the integral equation

$$U_k(t,s) = \begin{cases} E + \int\limits_s^t A(\xi)U_k(\xi,s)d\xi \\ + \sum\limits_{i=1}^{m} \int\limits_s^t D_i(\xi)U_k(\xi,s)\,dv_{i_k}(\xi) + \\ + \int\limits_s^t B(\xi)U_k(\xi-\tau,s)d\xi,\ t \ge s, \\ 0,\ s-\tau \le t < s. \end{cases}$$

and the pointwise limit of the sequence $U_k(t,s)$. Furthermore, the function $U(t,s)$ will be the solution of integral equation (9) by theorem 2.

5 Conclusion

We have introduced the definition of the solution for the system of differential equations with distributions in the right hand side of system and time delay. The entered definition possesses property of a physical realizability and it is natural from the point of view of the theory of control. Sufficient conditions for existence so defined solution are proved. The theorem about continuous dependence on the initial function of discontinuous solutions is proved. For the linear system with a matrix containing distributions the Cauchy formula is obtained.

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