Talbot effect obtained from complex transmittances

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Abstract: - The self-imaging position (or Talbot effect) for generalized complex functions, obtained as a product of scaled periodic components, is studied in this paper. The self-images of each periodic component and for the complete transmittance are shown. Different configurations including mixed gratings are considered and it is shown that the positions of Talbot planes are also related with the scaling factor.

Key-Words: Image processing, optical signals, Talbot effect, self-images, complex functions, periodic functions, Cantor functions, Walsh functions, cosenoidal functions

1 Introduction

In some previous works we have developed the mathematical methods that allow us to represent fractal gratings through a product superposition of periodic components [1-3]. For a simplification, in the present study these types of fractal structures have the following characteristics: 1) The periodic components have an aperture ratio α≤5, 2) The scaling factor (s) is an integer value and, to achieve fractal structures, it must comply with s>2, 3) Due to the conditions on self-imaging formation we use "periodic Cantor transmittances", this mean that the Cantor structure are periodically repeated and included into the initial periodic component.

The results that we present here are referred to the analysis, of the intensity field and the self-imaging formation [3-7] (or Talbot effect) when periodic components are superimposed as a product to obtain a total transmittance with complex geometry. This way, we show the influence of each component and the position of self-images of the total transmittance.

2 Field in the Fresnel region

According to the scalar theory, for an electromagnetic field with wavelength λ, the corresponding diffracted field in the Fresnel region from a transmittance $T(x)$, is given by:

$$U(x) = \mathrm{F}\left\{\tilde{T}(u)\, exp\left[-i\pi u^2 \lambda z\right]\right\} \quad (1)$$

where x is the transversal coordinate to the propagation direction z. The operators $\mathrm{F}\{\}$ and $\tilde{T}(u)$ indicate the Fourier transform and u is the variable in the Fourier space (or spatial frequency). The mathematical expressions for representing, through the Fourier analysis, an infinite periodic transmittance with rectangular basic elements, distributed with period D, and the corresponding Fourier transform, are given by:

$$T(x) = \sum_{n=0}^{\infty} C_n\, exp\left(2\pi i \frac{nx}{D}\right)$$

$$\tilde{T}(u) = \sum_{n=0}^{\infty} C_n\, \delta\left[u - \frac{n}{D}\right] \quad (2)$$

2.1 Self-imaging formation

From Eqs. (1) and (2), the electromagnetic field for this case is expressed as:

$$U(x) = \sum_{n=1}^{\infty} C_n\, exp\left(2\pi i \frac{nx}{D}\right) exp\left(-i\pi \frac{n^2}{D^2} \lambda z\right) \quad (3)$$

where it can be seen that, for the positions:

$$\pi \frac{n^2}{D^2} \lambda z = 2q \quad (\text{constant}) \Rightarrow z_T = 2q \frac{D^2}{\lambda} \quad (4)$$

there are replicas of the transmittance function defined in Eq. (2), because in this case the term into the second exponential is equal to one. Furthermore, z_T is the position for the first Talbot plane. It is clear that, if $q \in \Re$ (real, not integer) we are not on a Talbot plane (the second exponential in Eq. (3) is not equal to one).

3 Generalized transmittances with periodic functions

In previous work, we have shown the possibility of building fractal structures from binary functions or domains with periodic components [1-3]. These examples contain also a scaling factor s used for obtaining the successive scaled periodic components R_k from an initial one. Fractal transmittances with periodic components can be expressed as the product:

$$C(x) = \prod_{k=0}^{N} R_k(x; s^k) \tag{5}$$

being N the order of the structure and s the scaling factor. Complex functions using Eq. (5) is, in this way, very simple and important, from the optical point of view, when using periodic functions. Some examples included in this formulation are:

- Walsh functions [8]:

$$C(x) = \prod_{k=0}^{N} \left\{ \sum_{n=0}^{s^k-1} rect\left[\frac{x-x_n}{\Delta}\right] cos\left[\frac{2\pi}{b} n\right] \right\}$$

$$N = g_0\, s^0 + g_1\, s^1 + \cdots + g_{N-1}\, s^{N-1}$$

- Mixed gratings:

$$C(x) = \sum_{n_1=0}^{\infty} C_{n_1}\ exp\left(2\pi i \frac{n_1 x}{D_1}\right) \cdots \sum_{n_N=0}^{\infty} C_{n_N}\ exp\left(2\pi i \frac{n_N x}{D_N}\right)$$

- Cantor functions (see also [9]):

$$C(x) = \prod_{k=0}^{N} \left\{ \sum_{n_k=0}^{\infty} C_{n_k}\ exp\left(2\pi i \frac{n_k\, s^k x}{D}\right) \right\}$$

- Cosenoidal functions:

$$C(x) = \prod_{k=0}^{N} cos^2\left(\frac{2\pi}{D} s^k\ x\right)$$

4 Calculation of field for complex transmittances

Different theoretical methods and approaches have been developed to study the properties of self-imaging phenomenon. The Fourier analysis allows the development of theoretical calculation with applications for optical information processing. Now, including the expressions for the complex gratings into Eq. (1), we obtain the result:

$$U(x) = \sum_{n_0=1}^{\infty} C_{n_0} \cdots \sum_{n_N=1}^{\infty} C_{n_N}\ exp\left(-i\pi\lambda z \left(\sum_{k=0}^{N} \frac{n_k}{d_k}\right)^2\right)$$
$$exp\left(2\pi i\, x \left(\sum_{k=0}^{N} \frac{n_k}{d_k}\right)\right) \quad , \quad d_k = \frac{D}{s^k} \tag{6}$$

and, similarly to the result of Eqs. (3) and (4), the self-images position can be obtained if the following condition is accomplished:

$$exp\left[-i\pi\lambda z \left(\sum_{n_k=-\infty}^{+\infty} \frac{n_k}{d_k}\right)^2\right] = \text{constant}$$
$$\text{then} \quad U(x) = \prod_{i=0}^{N} R_i(x) \tag{7}$$

This fact can be seen in Fig. 1, that show the Montgomery rings in the spatial frequencies [8], for the case of Cantor transmittances with fractal dimension D≈0.6309 and D=0.5. Dot and dash lines are the self-imaging position for scaled periodic components and solid lines for the Cantor transmittance, with different orders.

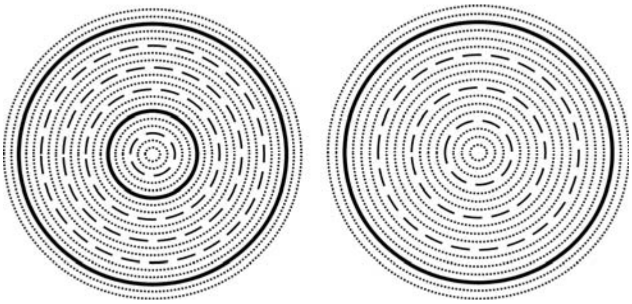

Fig. 1 – Montgomery rings for Cantor gratings.

Then, at any arbitrary position, where the periodic component with order j have a self-image, the condition of Eq. (7) becomes to:

$$\lambda z \left(\sum_{n_k=0}^{\infty} \frac{n_k}{d_k}\right)^2 = \lambda z \left[\left(\frac{n_j}{d_j}\right)^2 + 2\left(\frac{n_j}{d_j}\right)\left(\sum_{\substack{n_k=0 \\ k\neq j}}^{\infty} \frac{n_k}{d_k}\right) + \left(\sum_{\substack{n_k=0 \\ k\neq j}}^{\infty} \frac{n_k}{d_k}\right)^2 \right] \tag{8}$$

$$\frac{\lambda z}{D^2} s^j \left[n_j s^j + 2 n_j \left(\sum_{\substack{n_k=0 \\ k \neq j}}^{\infty} n_k s^k \right) \right] = q_j \qquad \text{(integer)} \tag{9}$$

Some experimental results obtained with a complex transmittance are shown in Fig. 2. The positions of self-images of two components (j_1 and j_2) have also related through:

$$\frac{q_1}{q_2} = s^{j_1 - j_2} \tag{10}$$

Then, the self-imaging for the complete fractal set is obtained at the position of self-imaging of the periodic component with the bigger period. This is:

$$Z_T[C] = \frac{2 q D^2}{\lambda} \qquad \text{with} \quad D = s^N d_N \tag{11}$$

Where D is the period of such component and d_N is the period for the component of order N.

5 Conclusions

The self-imaging phenomenon, for generalized complex functions, obtained as a product superposition of periodic components, is studied. In this work, the periodic functions can have real or integer scaling factors. The self-image positions are independent for each periodic component and are expressed as a function of the scaling factor. For a certain distance along the propagation, we can see in the mathematical expressions that a self-image of the total fractal structure is achieved.

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Acknowledgments

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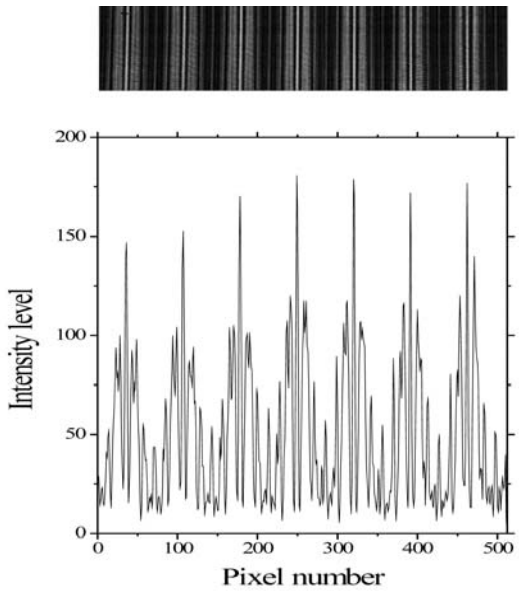
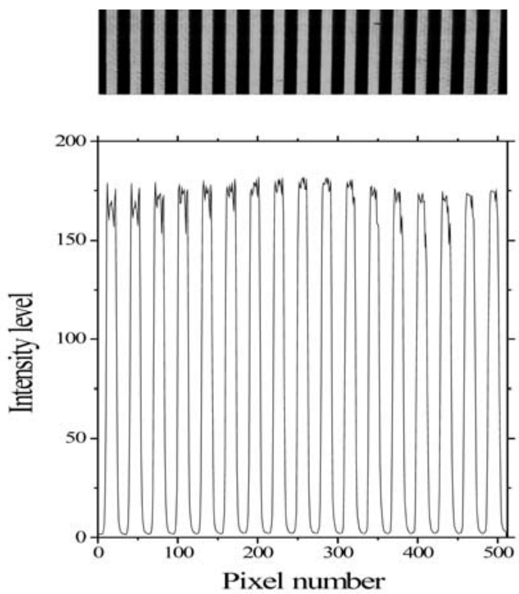


(a)



(b)

Fig. 2 – (a) Complex grating image at the Talbot plane for one periodic component (b).