Intuitionistic Fuzzy Numbers and It's Applications in Fuzzy Optimization Problem

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Abstract: - In this paper we introduce the trapezoidal intuitionistic fuzzy numbers and prove some operation for them. Also we introduce the intuitionistic fuzzy optimization problem by application of the membership and non-membership functions.

Key-words: - Fuzzy number, Intuitionistic fuzzy set, Trapezoidal fuzzy number

1 Introduction

In our daily life moments, we frequently deal with vague or imprecise information. Information available is sometimes vague, sometimes inexact or sometimes insufficient. Out of several higher order fuzzy sets, intuitionistic fuzzy sets(IFS) [2,3] have been found to be highly useful to deal with vagueness. There are situations where due to insufficiency in the information available, the evaluation of membership values is not possible up to our satisfacation. Due to the some reason evaluation of non-membership values is not also always possible and consequently there remains a part indeterministic on which hesitation survives. Certainly fuzzy sets theory is not appropriate to deal with such problem, rather intuitionistic fuzzy set(IFS) theory is more suitable. In this paper we introduce the definition of intuitionistic fuzzy numbers (IFN) and make some operations on IF numbers.

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2 Preliminaries

This section is divided into two subsection. First, we recollect some relevant basic preliminaries, in particular, the work of Atanassov [2,3]. In the second subsection we review some basic definition of fuzzy numbers.

2.1 Intuitionistic fuzzy sets

Let X be the universal set. In the following, we will describe those aspects of intuitionistic fuzzy sets which will be needed in our next discussion.

Definition 2.1.1 An IFS A in X is given by [2,3]

$$A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\},$$

where the functions $\mu_A, \nu_A : X \to [0,1]$ define respectively, the degree of membership and degree of non-membership of the element $x \in X$ to the set A, which is a subset of X, and for every $x \in X$, $0 \leq \mu(x) + \nu(x) \leq 1$. Obviously, every fuzzy set has the form

$$\{(x, \mu_A(x), \mu_{A^c}(x)) \mid x \in X\}.$$

For each IFS A in X, we will call

$$\Pi_A(x) = 1 - \mu(x) - \nu(x)$$

the intuitionistic fuzzy index of x in A. It is obvious that $0 \leq \Pi_A(x) \leq 1,\ \forall x \in X$.

Definition 2.1.2 If A and B are two IF sets of the set X, then

•$A \subset B$ iff
$$\forall x \in X,\ [\mu_A(x) \leq \mu_B(x),\ \nu_A(x) \geq \nu_B(x)],$$
•$A \supset B$ iff $B \subset A$,
•$A = B$ iff
$$\forall x \in X,\ [\mu_A(x) = \mu_B(x),\ \nu_A(x) = \nu_B(x)],$$
•$\overline{A} = \{(x, \nu_A(x), \mu_A(x)) \mid x \in X\}$,
•$A \cap B = \{(x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))) \mid x \in X\}$,
•$A \cup B = \{(x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))) \mid x \in X\}$,
•$A + B = \{(x, \mu_A(x) + \mu_B(x) - \mu_A(x).\mu_B(x), \nu_A(x).\nu_B(x)) \mid x \in X\}$,
•$A.B = \{(x, \mu_A(x).\mu_B(x), \nu_A(x) + \nu_B(x) - \nu_A(x).\nu_B(x)) \mid x \in X\}$,
•$\Box A = \{(x, \mu_A(x), 1 - \mu_A(x)) \mid x \in X\}$,
•$\diamond A = \{(x, 1 - \nu_A(x), \nu_A(x)) \mid x \in X\}$,

From definition of the product of two IF sets, we see that

$$A^2 = \{(x, [\mu_A(x)]^2, 1 - (1 - \nu_A(x))^2) \mid x \in X\},$$

$$A^3 = \{(x, [\mu_A(x)]^3, 1 - (1 - \nu_A(x))^3) \mid x \in X\},$$

and in general, for any positive integer n,

$$A^n = \{(x, [\mu_A(x)]^n, 1 - (1 - \nu_A(x))^n) \mid x \in X\}.$$

It can be easily verified that

$$0 \leq [\mu_A(x)]^n + [1 - (1 - \nu_A(x))^n] \leq 1,$$

even if n is any positive real number. Thus the IF set A^n defined above is an IF set for any positive real number n.

2.2 Fuzzy numbers

The most important subfamily of all fuzzy sets are fuzzy numbers. It is not surprising since the predominant carrier of information are numbers. The notion of a fuzzy number was introduced by Dubois and Prade [4,5].

Definition 2.2.1 A fuzzy set A of the real line $I\!R$ with membership function $\mu_A : I\!R \to [0,1]$ is called a fuzzy number if
a) A is normal, i.e. there exist an element x_0 such that $\mu_A(x_0) = 1$,
b) A is fuzzy convex, i.e. $\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \mu_A(x_1) \wedge \mu_A(x_2)\ \ \forall x_1, x_2 \in I\!R,\ \forall \lambda \in [0,1]$,
c) μ_A is upper semicontinuous,
d) *supp* A is bounded.

It is known that for any fuzzy number A, there exist four numbers $a_1,\ a_2,\ a_3,\ a_4 \in I\!R$ and two functions $f_A,\ g_a : I\!R \to [0,1]$, where f_A is nondecreasing and g_A is nonincreasing, such that we can describe a membership function μ_A in a following manner

$$\mu_A(x) = \begin{cases} 0 & \text{if} \quad x < a_1, \\ f_A(x) & \text{if} \quad a_1 \leq x \leq a_2, \\ 1 & \text{if} \quad a_2 \leq x \leq a_3, \\ g_A(x) & \text{if} \quad a_3 \leq x \leq a_4, \\ 0 & \text{if} \quad a_4 < x. \end{cases}$$

Functions f_A and g_a are called the left side and the right side of a fuzzy number A, respectively.
A useful tool for dealing with fuzzy numbers are their α-cuts. The α-cut of a fuzzy number A is a nonfuzzy set defined as

$$A_\alpha = \{x \in I\!R | \mu_A(x) \geq \alpha\}.$$

According to the definition of a fuzzy number it is seen at once that every α-cut of a fuzzy number is a closed interval. Hence we have $A_\alpha = [A_L(\alpha), A_U(\alpha)]$, where

$$A_L(\alpha) = \inf\{x \in I\!R | \mu_A(x) \geq \alpha\},$$

$$A_U(\alpha) = \sup\{x \in I\!R | \mu_A(x) \geq \alpha\},$$

Another important notion connected with fuzzy numbers is an expected interval $EI(A)$ of a fuzzy number A, introduced independently by Dubios and Prade [4] and Heilpern [7]. It is given by

$$EI(A) = [E_*(A), E^*(A)]$$

$$= [\int_0^1 A_L(\alpha)d\alpha, \int_0^1 A_U(\alpha)d\alpha)].$$

It can be shown that if A is a fuzzy number with continuous and strictly monotonic sides f_A and g_A then

$$E_*(A) = a_2 - \int_{a_1}^{a_2} f_A(x)dx,$$

$$E^*(A) = a_3 - \int_{a_3}^{a_4} g_A(x)dx.$$

3 Intuitionistic fuzzy numbers

In this section first we review some definition from [6], then we introduce the trapezoidal intuitionistic fuzzy numbers and prove some properties for these fuzzy numbers.

Definition 3.1 An IFS $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in I\!R\}$ of the real line is called an intuitionistic fuzzy number(IFN) if
a) A is IF-normal, i.e. there exist at least two points $x_0,\ x_1 \in X$ such that $\mu_A(x_0) = 1$, and $\nu_A(x_1) = 1$,
b) A is IF-convex, i.e. its membership function μ is fuzzy convex and its nonmembership function ν is fuzzy concave,
c) μ_A is upper semicontinuous and ν_A is lower semicontinuous,
d) *supp* $A = \{x \in X | \nu_A(x) < 1\}$ is bounded.

From the definition given above we get at once that for any IFN A there exist eight numbers $a_1,\ a_2,\ a_3,\ a_4,\ b_1,\ b_2,\ b_3,\ b_4 \in I\!R$ such that $b_1 \le a_1 \le b_2 \le a_2 \le a_3 \le b_3 \le a_4 \le b_4$ and four functions $f_A, g_A, h_A.k_A : I\!R :\to [0,1]$, called the sides of a fuzzy number, where f_A and k_A are nondecreasing and g_A and h_A are nonincreasing, such that we can describe a membership function μ_A in form

$$\mu_A(x) = \begin{cases} 0 & \text{if} \quad x < a_1, \\ f_A(x) & \text{if} \quad a_1 \le x \le a_2, \\ 1 & \text{if} \quad a_2 \le x \le a_3, \\ g_A(x) & \text{if} \quad a_3 \le x \le a_4, \\ 0 & \text{if} \quad a_4 < x, \end{cases}$$

while a nonmembership function ν_A has a following form

$$\nu_A(x) = \begin{cases} 0 & \text{if} \quad x < b_1, \\ h_A(x) & \text{if} \quad b_1 \le x \le b_2, \\ 1 & \text{if} \quad b_2 \le x \le b_3, \\ k_A(x) & \text{if} \quad b_3 \le x \le b_4, \\ 0 & \text{if} \quad b_4 < x. \end{cases}$$

It is worth noting that each IFN $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in I\!R\}$ is a conjunction of two fuzzy numbers: A^+ with a membership function $\mu_{A^+}(x) = \mu_A(x)$ and A^- with a membership function $\mu_{A^-}(x) = 1 - \nu_A(x)$. It is seen that *supp* $A^+ \subseteq$ *supp* A^-.
In the case of intuitionistic fuzzy numbers it is convenient to distinguish following α-cuts: $(A^+)_\alpha$ and $(A^-)_\alpha$. It is easily seen that

$$(A^+)_\alpha = \{x \in I\!R | \mu_A(x) \ge \alpha\} = A_\alpha,$$

$$(A^-)_\alpha = \{x \in I\!R | 1 - \nu_A(x) \ge \alpha\}$$
$$= \{x \in I\!R | \nu_A(x) \le 1 - \alpha\} = A^{1-\alpha}.$$

According to the definition it is seen at once that every α-cut, $(A^+)_\alpha$ or $(A^-)_\alpha$ is a closed interval. Hence we have $(A^+)_\alpha = [A_L^+(\alpha), A_U^+(\alpha)]$ and $(A^-)_\alpha = [A_L^-(\alpha), A_U^-(\alpha)]$, respectively, where

$$A_L^+(\alpha) = \inf\{x \in I\!R | \mu_A(x) \ge \alpha\},$$

$$A_U^+(\alpha) = \sup\{x \in I\!R | \mu_A(x) \ge \alpha\},$$

$$A_L^-(\alpha) = \inf\{x \in I\!R | \nu_A(x) \le 1 - \alpha\},$$

$$A_U^-(\alpha) = \sup\{x \in I\!R | \nu_A(x) \le 1 - \alpha\}.$$

If the sides of the fuzzy numbers A are strictly monotone then, we my adopt the convention that $f_A^{-1}(\alpha) = A_L^+(\alpha)$, $g_A^{-1}(\alpha) = A_U^+(\alpha)$, $h_A^{-1}(\alpha) = A_L^-(\alpha)$ and $k_A^{-1}(\alpha) = A_U^-(\alpha)$.
In particular, if the decreasing functions f_A and k_A and increasing functions g_A and h_A be linear then we will have the trapezoidal intuitionistic fuzzy numbers (TIFN).

Definition 3.2 A is a trapezoidal intuitionistic fuzzy number with parameters $b_1 \le a_1 \le b_2 \le a_2 \le a_3 \le b_3 \le a_4 \le b_4$ and denoted by

$A = (b_1, a_1, b_2, a_2, a_3, b_3, a_4, b_4)$. In this case we will give

$$\mu_A(x) = \begin{cases} 0 & \text{if} \quad x < a_1, \\ \frac{x-a_1}{a_2-a_1} & \text{if} \quad a_1 \le x \le a_2, \\ 1 & \text{if} \quad a_2 \le x \le a_3, \\ \frac{x-a_4}{a_3-a_4} & \text{if} \quad a_3 \le x \le a_4, \\ 0 & \text{if} \quad a_4 < x, \end{cases}$$

and

$$\nu_A(x) = \begin{cases} 0 & \text{if} \quad x < b_1, \\ \frac{x-b_2}{b_1-b_2} & \text{if} \quad b_1 \le x \le b_2, \\ 1 & \text{if} \quad b_2 \le x \le b_3, \\ \frac{x-b_3}{b_4-b_3} & \text{if} \quad b_3 \le x \le b_4, \\ 0 & \text{if} \quad b_4 < x. \end{cases}$$

If in a TIFN A, we let $b_2 = b_3$ (and hence $a_2 = a_3$), we will give a triangular intuitionistic fuzzy number (TrIFN) with parameters $b_1 \le a_1 \le b_2 (= a_2 = a_3 = b_3) \le a_4 \le b_4$ and denoted by $A = (b_1, a_1, b_2, a_4, b_4)$.

We can prove the following properties for trapezoidal (triangular) intuitionistic fuzzy numbers.

Lemma 3.3 If $A = (b_1, a_1, b_2, a_2, a_3, b_3, a_4, b_4)$ and $B = (b'_1, a'_1, b'_2, a'_2, a'_3, b'_3, a'_4, b'_4)$ be two trapezoidal intuitionistic fuzzy numbers, then $A + B$ also is a trapezoidal intuitionistic fuzzy number and $A + b = (b_1 + b'_1, a_1 + a'_1, b_2 + b'_2, a_2 + a'_2, a_3 + a'_3, b_3 + b'_3, a_4 + a'_4, b_4 + b'_4)$.

Lemma 3.4 If $A = (b_1, a_1, b_2, a_2, a_3, b_3, a_4, b_4)$ be a trapezoidal intuitionistic fuzzy number and r be a real number then rA is a trapezoidal intuitionistic fuzzy number and

$rA=$

$$\begin{cases} (rb_1, ra_1, rb_2, ra_2, ra_3, rb_3, ra_4, rb_4) & \text{if} \;\; r > 0, \\ (rb_4, ra_4, rb_3, ra_3, ra_2, rb_2, ra_1, rb_1) & \text{if} \;\; r < 0. \end{cases}$$

The expected interval of an intuitionistic fuzzy number $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in I\!R\}$ is a crisp interval $\widetilde{EI}(A)$ given by (see [6])

$$\widetilde{EI}(A) = [\widetilde{E_*}(A), \widetilde{E^*}(A)]$$

where

$$\widetilde{E_*}(A) = \frac{b_1 + a_2}{2} + \frac{1}{2}\int_{b_1}^{b_2} h_A(x)dx - \frac{1}{2}\int_{a_1}^{a_2} f_A(x)dx,$$

$$\widetilde{E^*}(A) = \frac{a_3 + b_4}{2} + \frac{1}{2}\int_{a_3}^{a_4} g_A(x)dx - \frac{1}{2}\int_{b_3}^{b_4} k_A(x)dx.$$

4 Intuitionistic fuzzy Optimization

In general, an optimization problem includes objective(s) and constraints. In fuzzy optimization problems, fuzzy optimal control, linear programming with fuzzy parameters, etc. the objective(s) and/or constraints or parameters and relations are represented by fuzzy sets. These fuzzy sets explain the degree of satisfaction of the respective condition and are expressed by their membership functions.

Let us consider an fuzzy optimization problem:

$$\begin{array}{lll} \widetilde{\min}: & f_i(x) & i = 1, \ldots, p, \\ \text{s.t.} & g_j(x) \widetilde{\le} 0 & j = 1, \ldots, q, \end{array}$$

where $\widetilde{\min}$ denotes fuzzy minimization and $\widetilde{\le}$ denotes fuzzy inequality.

It is transformed via Bellman-Zadeh's Approach to the following optimization problem:

$$\begin{array}{lll} \max: & \mu_i(x) & i = 1, \ldots, p+q, \\ \text{s.t.} & 0 \le \mu_i(x) \le 1, x \in I\!R, & \end{array}$$

where $\mu_i(x)$ denotes degree of acceptance of x to the respective fuzzy sets.

In the case when the degree of rejection (nonmembership) is defined simultaneously with the degree of acceptance (membership) and when both these degrees are not complementary to each other then IF sets can be used as a more general and full tool for describing this uncertainty. It is possible to represent deeply existing nuances in problem formulation defining objective(s) and constraints (or part of them) by IF sets, i.e. by pairs of membership ($\mu_i(x)$) and rejection ($\nu_i(x)$) function. An intuitionistic fuzzy optimization (IFO) problem is formulated as follows:

$$\begin{array}{lll} \max: & \mu_i(x) & i = 1, \ldots, p+q, \\ \min: & \nu_i(x) & i = 1, \ldots, p+q, \\ \text{s.t.} & \nu_i(x) \le 0 & i = 1, \ldots, p+q, \\ & \mu_i(x) \ge \nu_i(x) & i = 1, \ldots, p+q, \\ & \mu_i(x) + \nu_i(x) \le 1 & i = 1, \ldots, p+q, \end{array}$$

where $\mu_i(x)$ denotes the degree of membership of x to the ith IF set and $\nu_i(x)$ denotes the degree of non-membership of x from the ith IF set.

Based on the Bellman-Zadeh's approach for fuzzy optimization problem, we can transform this IFO problem to a classical optimization problem as follow:

$$\begin{array}{lll} \max: & \alpha - \beta, & \\ \text{s.t.} & \mu_i(x) \geq \alpha & i = 1, \ldots, p+q, \\ & \nu_i(x) \leq \beta & i = 1, \ldots, p+q, \\ & \alpha \geq \beta, \quad \beta \geq 0, & \alpha + \beta \leq 1. \end{array}$$

5 Conclusion

In this paper we have introduced the definition of intuitionistic fuzzy numbers. We have also proposed the concept of Trapezoidal (triangular) intuitionistic fuzzy numbers and we have proved some operation for them. We have also proposed a method for intuitionistic fuzzy optimization problem.

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