Confidence Interval for Fuzzy Process Capability Index

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Abstract: - Capability indices compare the actual performance of a manufacturing process to the desired performance. In practice these indices are estimated using sample data, often with quite small sample sizes. Thus, it is of interest to obtain confidence limits for actual capability index given a sample estimate. Most of the traditional methods for assessing the capability of manufacturing process are dealt with crisp quality. In this paper we obtain $100(1-\alpha)\%$ fuzzy confidence interval for $\tilde{C}_p$ fuzzy process capability index, where instead of precise quality we have two membership functions for specification limits. A numerical example is given to clarify the method.

Key-words:- Fuzzy process capability index; Triangular fuzzy number; Ranking function; Fuzzy confidence interval.

1 Introduction and Preliminaries

A process capability index (PCI) is a real number as a summary that compares the behaviour of a product or process characteristic to engineering specifications. This measure is also called performance index. A process is said to be capable if with high probability the real valued quality characteristic of the produced items lies between the lower and upper specification limits [9].

There are several PCIs such as C_p, C_{pk}, C_{pm} and so on, which are used to estimate the capability of a manufacturing process where in most cases the normal distribution and a large sample size is assumed for population of data [9, 10].

After the inception of the notion of fuzzy sets by Zadeh [23] there are efforts by many authors to apply this notion in statistics. For these trends one can see Taheri [20].

In some cases specification limits (SLs) are not precise numbers and they are expressed in fuzzy terms, so that the classical capability indices could not be applied. For such cases Yongting [22] introduced a process capability index C_p as a real number and it was used by Sadeghpour-Gildeh [19]. Lee investigated a process capability index, C_{pk}, as a fuzzy set [13]. Parchami et al. introduced fuzzy PCIs as fuzzy numbers and discussed relations that governing between them when SLs are fuzzy rather

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than crisp [17]. The organization of this paper is as follows. In Section 2, we review some traditional PCIs and their confidence intervals. In Section 3, we represent fuzzy PCIs and then we review ranking functions in Section 4. In Section 5 we will obtain fuzzy confidence interval for a new fuzzy PCI and we will present a numerical example. The final section is the conclusion part.

Let R be the set of real numbers. Set:

$F(\mathsf{R}) = \{A \mid A : \mathsf{R} \to [0,1],\ A \text{ is a continuous function}\}$,

$F_T(\mathsf{R}) = \{T_{a,b,c} \mid a,b,c \in \mathsf{R}, a \le b \le c\}$,

where

$$T_{a,b,c}(x) = \begin{cases} (x-a)/(b-a) & \text{if} \quad a \le x < b, \\ (c-x)/(c-b) & \text{if} \quad b \le x < c, \\ 0 & \text{elsewhere.} \end{cases} \quad (1)$$

Any $A \in F(\mathsf{R})$ is called a fuzzy set on R and any $T_{a,b,c} \in F_T(\mathsf{R})$ is called a fuzzy triangular number, which we sometimes write as $T(a,b,c)$. We assume $T(a,a,a)$ be $I_{\{a\}}$, the indicator function of a.

The following definition could be given by using the extension principle, see [16].

Definition 1.1 Let $T(a,b,c) \in F_T(\mathsf{R})$, $k \in \mathsf{R}$, $k \ge 0$. Define the operation $\otimes$ on $F_T(\mathsf{R})$ as follows

$$T(a,b,c) \otimes k = T(ka, kb, kc), \quad (2)$$

called the *multiplication* of $T(a,b,c)$ by k.

2 Traditional PCIs and Confidence Intervals

The process capability compares the output of a process to the SLs by using capability indices. Frequently, this comparison is made by forming the ratio of the width between the process SLs to the width of the natural tolerance limits, as measured by 6 process standard deviation units. This method leads to make a statement about how well the process meets specifications [15]. Several PCIs are introduced in the literature such as C_p, C_{pk}, C_{pm} and so on, see [9, 10]. For convenience, we will denote the upper and lower specification limits by U and L, respectively, rather than the more customary USL and LSL notations. When univariate measurements are concerned, we will denote the corresponding random variate by X. The expected value and standard deviation of X will be denoted by μ and σ, respectively. We will limit ourselves to the situation where μ is in the specification interval, i.e. $L \le \mu \le U$, and we assume that the measured characteristic should have a normal distribution (at least, approximately).

The commonly recognized PCIs are

$$C_p = \frac{U-L}{6\sigma} = \frac{w}{6\sigma}, \quad (3)$$

where $w = U - L$. This C_p is used when $\mu = M$, where $M = (U+L)/2$.

$$C_{pk} = \frac{w - 2|\mu - M|}{6\sigma} = \frac{\min\{U-\mu, \mu - L\}}{3\sigma}, \quad (4)$$

and

$$C_{pm} = \frac{w}{6\sqrt{\sigma^2 + (\mu - T)^2}} = \frac{w}{6\sqrt{E[(X-T)^2]}}, \quad (5)$$

where T is target value and $E[.]$ denotes expected value. For each of these indices a large value implies a better distribution of the quality characteristic.

Introduction of C_p is ascribed to Juran [6]; that of C_{pk} to Kane [8]; that of C_{pm} for the most part to Hsiang and Taguchi [5].

Substituting the sample mean and standard deviation in (3), (4) and (5) will provides a point estimate for any of these indices. We would never expect this point estimate to be exactly equal to the real value of the population parameter. So we often also compute a $100(1-\alpha)\%$ confidence interval for parameter. In practice a confidence bound can be used to guard against false optimism. In the following several confidence intervals for PCIs from Kotz and Johnson [10] are quoted.

Kane [8] suggest the $100(1-\alpha)\%$ formula for confidence interval limits of C_p

$$\left(\hat{C}_p \sqrt{\frac{\chi^2_{n-1,\alpha/2}}{(n-1)}}, \hat{C}_p \sqrt{\frac{\chi^2_{n-1,1-\alpha/2}}{(n-1)}} \right), \quad (6)$$

where $\hat{C}_p = \dfrac{U-L}{6\hat{\sigma}}$.

Kushler and Hurley [12] suggest the simple formula for lower bond of C_{pk}

$$\hat{C}_{pk}\left(1 - \frac{Z_{1-\alpha}}{(2n-2)^{1/2}}\right), \quad (7)$$

where $\hat{C}_{pk} = (U - L - 2|\bar{x} - M|)/6\hat{\sigma}$, and Kotz and Lovelace [11] reports that Dovich (1992) gives the corresponding approximate $100(1-\alpha)\%$ formula for confidence interval limits of C_{pk} as follows

$$\left(\hat{C}_{pk}\left(1-\frac{Z_{1-\alpha/2}}{(2n-2)^{1/2}}\right),\hat{C}_{pk}\left(1+\frac{Z_{1-\alpha/2}}{(2n-2)^{1/2}}\right)\right). \quad (8)$$

3 Fuzzy Process Capability Indices

As explained in Section 1, the C_p index based on fuzzy SLs is introduced as a real number by Yongting [22] and was used by other authors. But when we have fuzzy SLs, it would be more realistic to have a C_p which is also fuzzy, since a fuzzy capability index could be more informative rather than a precise number. In this situation Parchami et. al. [17] introduced PCIs as fuzzy numbers. They used fuzzy numbers such as $U(a_u,b_u,c_u)=T(a_u,b_u,c_u)\in F_T(\mathbb{R})$ and $L(a_l,b_l,c_l)=T(a_l,b_l,c_l)\in F_T(\mathbb{R})$ for engineering specification limits and they gave the following definitions.

Definition 3.1 A process with fuzzy specification limits, which we call a fuzzy process for short, is one which (approximately) satisfies the normal distribution condition [9].

Definition 3.2 Let $U(a_u,b_u,c_u),L(a_l,b_l,c_l)\in F_T(\mathbb{R})$ be the engineering fuzzy specification limits, where $a_u\geq c_l$. Then the new fuzzy PCIs are defined as follow

$$\tilde{C}_p=T\left(\frac{a_u-c_l}{6\sigma},\frac{b_u-b_l}{6\sigma},\frac{c_u-a_l}{6\sigma}\right), \quad (9)$$

note that $\tilde{C}_p$ is useful when $\mu=m$, where $m=\frac{b_u+b_l}{2}$. Also

$$\tilde{C}_{pk}=T\left(\frac{a_u-c_l-2|\mu-m|}{6\sigma},\frac{b_u-b_l-2|\mu-m|}{6\sigma},\frac{c_u-a_l-2|\mu-m|}{6\sigma}\right) \quad (10)$$

and

$$\tilde{C}_{pm}=T\left(\frac{a_u-c_l}{6\sqrt{\sigma^2+(\mu-T)^2}},\frac{b_u-b_l}{6\sqrt{\sigma^2+(\mu-T)^2}},\frac{c_u-a_l}{6\sqrt{\sigma^2+(\mu-T)^2}}\right), \quad (11)$$

where T is target value.

4 Ranking function

In the sequel sections we are going to give a fuzzy confidence interval, where comparing fuzzy numbers is an emergent and so an ordering approach is needed. We need a criterion for comparison of two fuzzy subsets. A simple but efficient approach for ordering of the elements of $F(\mathbb{R})$ is to define a ranking function $R:F(\mathbb{R})\to\mathbb{R}$ which maps each fuzzy number into the real line, where a natural order exists, see [14]. Define the order $\underset{R}{\leq}$ on $F(\mathbb{R})$ by

$$\tilde{A}\underset{R}{\geq}\tilde{B} \text{ iff } R(\tilde{A})\geq R(\tilde{B}),$$
$$\tilde{A}\underset{R}{\leq}\tilde{B} \text{ iff } R(\tilde{A})\leq R(\tilde{B}),$$
$$\tilde{A}\underset{R}{=}\tilde{B} \text{ iff } R(\tilde{A})=R(\tilde{B}),$$

where $\tilde{A}$ and $\tilde{B}$ are in $F(\mathbb{R})$.

Several ranking functions have been proposed by researchers to suit their requirements of the problems under consideration. For more details see [1, 21]. The Ranking function proposed by Roubens [4, 18] is defined by

$$R_r(\tilde{A})=\frac{1}{2}\int_0^1\left(inf\ \tilde{A}_\alpha+sup\ \tilde{A}_\alpha\right)d\alpha.$$

From now on, if R_r is the Roubens's ranking function, then we write $\underset{R_r}{\leq}$ simply as $\leq$.

One can easily prove the following lemmas.

Lemma 4.1 If $T(a,b,c)\in F_T(\mathbb{R})$, then the Roubens's ranking function reduces to following

$$R_r\left(T(a,b,c)\right)=\frac{2b+a+c}{4}. \quad (12)$$

Lemma 4.2 Let $m,n\in\mathbb{R}$, $T(a,b,c)\in F_T(\mathbb{R})$ and $2b+a+c\geq 0$. Then according the Roubens's ranking function

$$m\leq n \text{ iff } m\otimes T(a,b,c)\leq n\otimes T(a,b,c).$$

5 Fuzzy confidence intervals for fuzzy PCIs

Substituting the standard deviation in (9) will provides a point estimate for $\tilde{C}_p$, which we denote it by $\hat{\tilde{C}}_p$. Since $\hat{\tilde{C}}_p$, like other statistics, is subject to sampling variation, it is critical to compute a confidence interval to provide a range which includes the true $\tilde{C}_p$ with high probability.

Definition 5.1 Let $A, B \in F_T(\mathsf{R})$ and $A \le B$. The fuzzy interval $[A, B]$ is the set

$$[A, B] = \{C \in F_T(\mathsf{R}) \mid A \le C \le B\}.$$

Note that $[A, B]$ is nonempty, since $A, B \in [A, B]$.

Suppose that the set of all random samples of size n which are possible is $X^{(n)}$.

Definition 5.2 Any function $A : X^{(n)} \to F_T(\mathsf{R})$ is called a fuzzy statistic. Note that $A(X_1, \cdots, X_n)$ only depends on $X_1, \cdots, X_n$ and no any unknown parameters. When the observation $\mathbf{x} = (x_1, \cdots, x_n)$ is given, then the value of the statistic, $A(\mathbf{x})$ is just one triangular fuzzy number.

Let X be a measurable random variable on the probability space $(\Omega, \mathcal{F}, \Pr)$ and $T = T(a,b,c) \in F_T(\mathsf{R})$ such that $2b + a + c \ge 0$. We define

$$(X \otimes T)(\omega) = X(\omega) \otimes T, \ \forall \omega \in \Omega.$$

Then by Lemma 4.2, we can see that

$$\{\omega \in \Omega \mid k_1 \le X(\omega) \le k_2\} =$$

$$\{\omega \in \Omega \mid k_1 \otimes T \le X(\omega) \otimes T \le k_2 \otimes T\}. \quad (13)$$

Now we can give the following definition.

Definition 5.3 Let X be a measurable random variable on the probability space $(\Omega, \mathcal{F}, \Pr)$, $k_1, k_2 \in \mathsf{R}$ and $T = T(a,b,c) \in F_T(\mathsf{R})$ such that $2b + a + c \ge 0$. Then by (13) we define that

$$\Pr\left(k_1 \otimes T \underset{R_r}{\le} X \otimes T \underset{R_r}{\le} k_2 \otimes T\right) = 1 - \alpha,$$

if and only if

$$\Pr\left(k_1 \le X \le k_2\right) = 1 - \alpha,$$

and we say that $[A, B]$ is a $100(1-\alpha)\%$ fuzzy confidence interval for $X \otimes T$, where $A = k_1 \otimes T$ and $B = k_2 \otimes T$ are the observed fuzzy statistic as triangular fuzzy numbers.

Theorem 5.1 Suppose that $X_1, X_2, \cdots, X_n$ are independent, identically distributed random variables with $N(\mu, \sigma^2)$ and $U(a_u, b_u, c_u) \in F_T(\mathsf{R})$, $L(a_l, b_l, c_l) \in F_T(\mathsf{R})$ be the engineering fuzzy specification limits, where $a_u \ge c_l$. The following interval is a $100(1-\alpha)\%$ fuzzy confidence interval for $\tilde{C}_p$

$$\left(\hat{\tilde{C}}_p \sqrt{\frac{\chi^2_{n-1,\alpha/2}}{(n-1)}} \ , \ \hat{\tilde{C}}_p \sqrt{\frac{\chi^2_{n-1,1-\alpha/2}}{(n-1)}} \right),$$

where $\hat{\tilde{C}}_p = T\left(\dfrac{a_u - c_l}{6s}, \dfrac{b_u - b_l}{6s}, \dfrac{c_u - a_l}{6s}\right)$ and

$$s = \left\{ \frac{1}{n-1} \sum_{j=1}^{n} (x_j - \bar{x})^2 \right\}^{\frac{1}{2}}.$$

Remark 5.1 Let in a fuzzy process (1.4,4.5) be a 95% confidence interval for σ. A pictorial representation of 95% fuzzy confidence interval, drown by Maple software, for $\tilde{C}_p$ is given in the Fig.1.

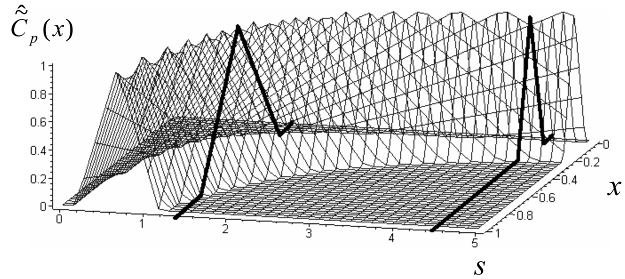

Fig.1. pictorial representation of 95% fuzzy confidence interval for $\tilde{C}_p$

Note that as s increases the $\hat{\tilde{C}}_p$ tends to a sharper fuzzy triangular number. The fact can be seen in the Fig.1 where the fuzzy confidence bounds are shown in bold.

Remark 5.2 Finding a fuzzy confidence interval for $\tilde{C}_{pk}$ is not as easy as for $\tilde{C}_p$. We will study such fuzzy intervals elsewhere.

Remark 5.3 When the process specification limits $U(a_u, b_u, c_u)$ and $L(a_l, b_l, c_l)$ are precise numbers and hence $a_u = b_u = c_u$ and $a_l = b_l = c_l$, in other words when they are indicator functions, then the introduced fuzzy confidence interval in Theorem 5.1 is a precise interval and it coincides to traditional confidence interval.

Now we apply our approach to find a fuzzy confidence interval for $\tilde{C}_p$.

Example 5.1 For a special product suppose that the specification limits are considered to be "approximately 4" and "approximately 8" which are characterized by $L(2,4,6) \in F_T(\mathsf{R})$ and $U(7,8,9)$

$\in F_T(\mathbb{R})$; respectively (see Fig.2). Assume that the process mean μ is 6 and the estimated process standard deviation is $2/3$.

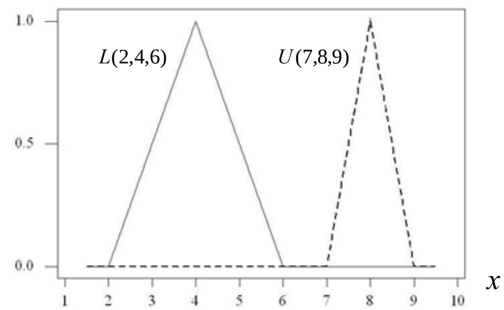

Fig.2. The membership function of fuzzy process specification limits in Example 5.1

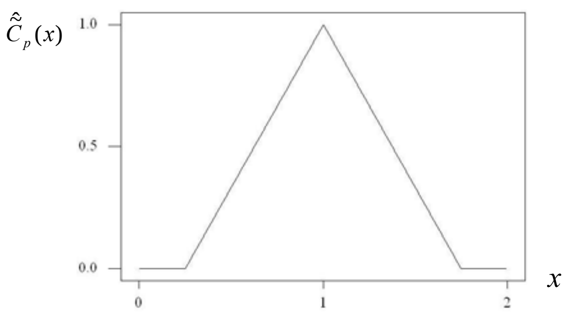

Fig.3. The membership function of fuzzy process capability index in Example 5.1

By (9) we can estimate $\tilde{C}_p$ with $\hat{\tilde{C}}_p = T(0.25,1,1.75)$. Hence, $\hat{\tilde{C}}_p$ is "approximately one", as shown in Fig.3.

Consider estimates of $\tilde{C}_p$ based on two samples from the same process, but with different sample size; 95% fuzzy confidence intervals for each sample have been calculated using the Theorem 5.1

$$T(0.14,0.55,0.96) \le \tilde{C}_p \le T(0.41,1.6,2.9),$$

for $n = 10$ and

$$T(0.17,0.69,1.21) \le \tilde{C}_p \le T(0.3,1.29,2.13),$$

for $n = 41$.

A large sample size results in more accuracy in the estimate, as seen in the tighter fuzzy confidence interval for the large sample size, see Fig.4. This example shows the danger of just looking at a point estimation only, without qualifying it via fuzzy confidence intervals.

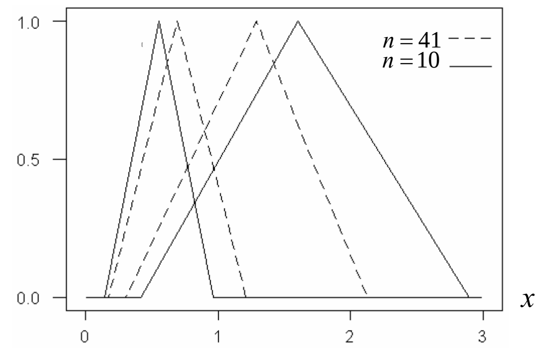

Fig.4. The membership function of fuzzy confidence intervals for $\tilde{C}_p$ index in Example 5.1

Our observation leads us to the following open problem.

Open problem: Let

$$\hat{\tilde{C}}_p = T\left(\frac{a_u - c_l}{6s}, \frac{b_u - b_l}{6s}, \frac{c_u - a_l}{6s}\right).$$

Is it true that :

$$\lim_{n\to\infty} \left(\hat{\tilde{C}}_p \sqrt{\frac{\chi^2_{n-1,\alpha/2}}{(n-1)}} \, , \, \hat{\tilde{C}}_p \sqrt{\frac{\chi^2_{n-1,1-\alpha/2}}{(n-1)}} \right) = \left\{ I_{\tilde{C}_p} \right\}.$$

6 Conclusion

If we define the specification limits (SLs) by fuzzy quantities, it is more appropriate to define the process capability indices as fuzzy numbers. Easily we can obtain point estimation for these fuzzy process capability indices, but we would never expect this point estimate to be exactly equal the parameter value, so we often also compute a $100(1-\alpha)\%$ confidence interval for parameter. In this paper we introduced a $100(1-\alpha)\%$ confidence interval for $\tilde{C}_p$ fuzzy process capability index, when the engineering SLs are triangular fuzzy numbers. A meaningful application of this new $100(1-\alpha)\%$ fuzzy confidence interval for $\tilde{C}_p$ is emerged and clarified by an example.

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