Relation Between Canonical and Optimized State Models of PWL Dynamical System Belonging to Class *C*

JIŘÍ POSPÍŠIL, ZDENĚK KOLKA, STANISLAV HANUS, VÁCLAV POSPÍŠIL
Department of Radio Electronics
Brno University of Technology
Purkyňova 118, 612 00 Brno
CZECH REPUBLIC
http://www.feec.vutbr.cz/UREL

Abstract: - Previously published two piecewise-linear (PWL) dynamical systems belonging to Class *C* with canonical and low eigenvalue sensitivity state models are compared. Their mutual relation is expressed analytically by similarity transformation conditions using the linear topological conjugacy and then illustrated graphically using the "*elastic space*" analogy.

Key-Words: Dynamical systems, state models, PWL systems

1 Introduction

Third-order piecewise-linear (PWL) dynamical systems belonging to Class *C* of vector fields in $\mathfrak{R}^3$ [1] can be described by the state equations in their matrix form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}\, h(\mathbf{w}^T\mathbf{x}) \ , \tag{1}$$

($\mathbf{A} \in \mathfrak{R}^{3x3}$, $\mathbf{b} \in \mathfrak{R}^3$, $\mathbf{w} \in \mathfrak{R}^3$) where PWL function

$$h(\mathbf{w}^T\mathbf{x}) = \tfrac{1}{2}\left(\left|\mathbf{w}^T\mathbf{x}+1\right| - \left|\mathbf{w}^T\mathbf{x}-1\right|\right) \tag{2}$$

is continuous, odd-symmetric and partitioning $\mathfrak{R}^3$ by two parallel planes into the inner (origin) region D_0 and two outer regions D_{+1}, D_{-1} (Fig. 1).

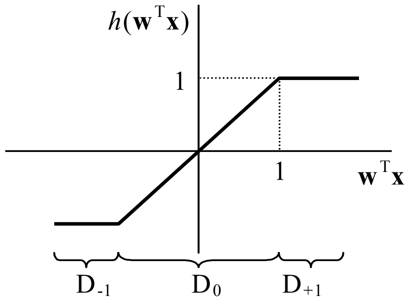

Fig. 1: Simple memoryless PWL feedback function.

The dynamical behavior of such systems is determined by two sets of eigenvalues representing two characteristic polynomials associated with the corresponding region [1], i.e.

$$D_0: \quad P(s) = \det(s\mathbf{1}-\mathbf{A}_0) = (s-\mu_1)(s-\mu_2)(s-\mu_3) = s^3 - p_1 s^2 + p_2 s - p_3 \tag{3}$$

$$D_{+1}, D_{-1}: \quad Q(s) = \det(s\mathbf{1}-\mathbf{A}) = (s-\nu_1)(s-\nu_2)(s-\nu_3) = s^3 - q_1 s^2 + q_2 s - q_3 \tag{4}$$

Symbol $\mathbf{1}$ represents the unity matrix, the state matrix associated with the inner region D_0 is given as

$$\mathbf{A}_0 = \mathbf{A} + \mathbf{b}\mathbf{w}^T \tag{5}$$

and coefficients p_i , q_i. (i = 1,2,3) are the equivalent eigenvalue parameters, which are related to the individual eigenvalues ν_i , μ_i (i = 1,2,3) as follows

$$p_1 = \mu_1 + \mu_2 + \mu_3, \qquad q_1 = \nu_1 + \nu_2 + \nu_3,$$
$$p_2 = \mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1, \qquad q_2 = \nu_1\nu_2 + \nu_2\nu_3 + \nu_3\nu_1,$$
$$p_3 = \mu_1\mu_2\mu_3, \qquad q_3 = \nu_1\nu_2\nu_3 \tag{6a,b,c}$$

Any two systems having the same eigenvalues are qualitatively equivalent and their mutual relations can be expressed by the linear topological conjugacy conditions [2] in the explicit form

$$\tilde{\mathbf{x}} = \mathbf{T}\mathbf{x} \ , \tag{7a}$$

$$\tilde{\mathbf{A}} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1} \ , \tag{7b}$$

$$\tilde{\mathbf{b}} = \mathbf{T}\mathbf{b} \ , \tag{7c}$$

where the complete transformation matrix is given as

$$\mathbf{T} = \tilde{\mathbf{K}}^{-1}\mathbf{K} \tag{7d}$$

Variables $\tilde{\mathbf{x}}$ and $\mathbf{x}$, state matrices $\tilde{\mathbf{A}}$ and $\mathbf{A}$, vectors $\tilde{\mathbf{b}}, \tilde{\mathbf{w}}$ and $\mathbf{b}, \mathbf{w}$ belong to the first and second systems, respectively. Partial transformation matrices $\tilde{\mathbf{K}}$ and $\mathbf{K}$ are defined by the nonsingular form [2]

$$\widetilde{\mathbf{K}}=\begin{bmatrix}\widetilde{\mathbf{w}}^T\\ \widetilde{\mathbf{w}}^T\widetilde{\mathbf{A}}\\ \widetilde{\mathbf{w}}^T\widetilde{\mathbf{A}}^2\end{bmatrix} \quad \text{and} \quad \mathbf{K}=\begin{bmatrix}\mathbf{w}^T\\ \mathbf{w}^T\mathbf{A}\\ \mathbf{w}^T\mathbf{A}^2\end{bmatrix} \qquad (8a,b)$$

fulfilling the observability condition of pairs $(\widetilde{\mathbf{A}},\widetilde{\mathbf{w}}^T)$ and $(\mathbf{A},\mathbf{w}^T)$, respectively.

Any state model topologically conjugated to Class C can be used as the reference model, i.e. any other model or system can be then expressed by this qualitatively equivalent reference form. Suitable models with very simple form of partial transformation matrix represent Chua's model corresponding to Chua's oscillator [1] and especially its first canonical ODE equivalent [3], [4].

In this contribution the state models of two previously published PWL dynamical systems belonging to Class C, i.e. the first canonical form [3], [4] and the optimized form with low eigenvalue sensitivity [6], [7], are defined and compared. Their mutual relation is expressed analytically by the similarity transformation [2], [5] utilizing linear topological conjugacy conditions (7a,b,c,d) and then illustrated graphically by using the "*elastic space*" analogy [4].

2 First canonical form

The first elementary canonical state model [3], [4] can be used as the basic initial system where the vectors and state matrix in eqn (1) are

$$\widetilde{\mathbf{x}}=\mathbf{x}_I=\begin{pmatrix}x_I\\ y_I\\ z_I\end{pmatrix}, \quad \widetilde{\mathbf{w}}=\mathbf{w}_I=\begin{pmatrix}1\\0\\0\end{pmatrix} \qquad (9a,b)$$

$$\widetilde{\mathbf{A}}=\mathbf{A}_I=\begin{bmatrix}q_1 & -1 & 0\\ q_2 & 0 & -1\\ q_3 & 0 & 0\end{bmatrix}, \quad \widetilde{\mathbf{b}}=\mathbf{b}_I=\begin{pmatrix}p_1-q_1\\ p_2-q_2\\ p_3-q_3\end{pmatrix} \qquad (9c,d)$$

Then the complete state equations can be expressed as

$$\left.\begin{aligned}\dot{x}_I &= q_1x_I - y_I + (p_1-q_1)\,h(x_I)\\ \dot{y}_I &= q_2x_I - z_I + (p_2-q_2)\,h(x_I)\\ \dot{z}_I &= q_3x_I + (p_3-q_3)\,h(x_I)\end{aligned}\right\} \qquad (10)$$

3 Sensitivity-optimized form

Recently published design procedure [6], represents the results optimized from the viewpoint of the minimum relative eigenvalue sensitivities. It provides the possibility to derive the general form of the corresponding optimization condition for the minimum sum of the relative eigenvalue sensitivity squares with respect to the change of the individual model parameters. The optimization procedure can be applied to the state models based on the block-decomposed state matrix [6], i.e. utilizing results for the second-order systems, the third-order model with upper block-triangular state matrix containing optimized second-order submatrix can be designed. Suppose one pair of the complex conjugate eigenvalues and one real eigenvalue in both outer and inner regions of the elementary PWL function (2), i.e. ($\nu_{1,2}=\nu'\pm j\nu''$, ν_3 - real ; $\mu_{1,2}=\mu'\pm j\mu''$, μ_3 - real).

The state matrix and the vectors in (1) have the form

$$\mathbf{x}=\mathbf{x}_{\mathbf{opt}}=\begin{pmatrix}x_o\\ y_o\\ z_o\end{pmatrix}, \qquad \mathbf{w}=\mathbf{w}_{\mathbf{opt}}=\begin{pmatrix}1\\ w_{o2}\\ 1\end{pmatrix}, \qquad (11a)$$

$$\mathbf{A}=\mathbf{A}_{\mathbf{opt}}=\left[\begin{array}{cc|c}\nu' & -\nu'' & -b_{o1}\\ \nu'' & \nu' & -b_{o2}\\ \hline 0 & 0 & \nu_3\end{array}\right], \quad \mathbf{b}_{\mathbf{opt}}=\begin{pmatrix}b_{o1}\\ b_{o2}\\ b_{o3}\end{pmatrix}, \qquad (11b)$$

where $b_{o1}=\mu'-\nu'$, $\quad b_{o3}=(\mu_3-\nu_3)$, $\qquad$ (11c)

while the parameters b_{o2} and w_{o2} are [6]

$$b_{o2}=\frac{(\mu'-\nu')^2}{\nu''-\mu''\mathrm{K}}, \qquad w_{o2}=\frac{\nu''-\mu''\mathrm{K}}{\mu'-\nu'}. \qquad (11d)$$

Then the state matrix associated with the inner region has the lower block-triangular form

$$\mathbf{A}_0=\mathbf{A}_{0\,\mathbf{opt}}=\left[\begin{array}{cc|c}\mu' & -\mu''\mathrm{K} & 0\\ \mu''\mathrm{K}^{-1} & \mu' & 0\\ \hline b_{o3} & b_{o3}\,w_2 & \mu_3\end{array}\right]. \qquad (12)$$

The optimizing coefficient K is expressed as the real root of the quadratic equation [6]

$$\mathrm{K}^2-2\mathrm{K}(\mathrm{M}+1)+1=0, \text{ i.e. } \mathrm{K}=1+\mathrm{M}\pm\sqrt{\mathrm{M}(\mathrm{M}+2)}, \qquad (13)$$

where the auxiliary parameter M is given in the form

$$\mathrm{M}=\frac{(\mu'-\nu')^2+(\mu''-\nu'')^2}{2\mu''\nu''}>0, \quad (\mu'',\nu''\neq 0).$$

By such a way the model having very low eigenvalue sensitivities in both the outer and inner regions of the PWL feedback function is obtained. The complete state equations of the optimized third-order PWL autonomous system can be rewritten into the form

$$\begin{aligned}\dot{x}_o &= \nu'\left[x_o+y_o-h\,(x_o+w_{o2}\,y_o+z_o)\right]-\nu''y_o+\\ &\quad+\mu'\left[h\,(x_o+w_{o2}\,y_o+z_o)-z_o\right]\\ \dot{y}_o &= \nu''x_o+\nu'y_o+b_{o2}\left[h\,(x_o+w_{o2}\,y_o+z_o)-z_o\right]\end{aligned} \qquad (14)$$

$$\dot{z}_o = \nu_3 \lfloor z_o - h(x_o + w_{o2} y_o + z_o) \rfloor + \\ + \mu_3 \, h(x_o + w_{o2} y_o + z_o)$$

where the basic individual parameters are separated. The corresponding integrator-based block diagram, which is suitable as the prototype [6] for the practical realization of the optimized chaotic oscillator, is shown in Fig. 2.

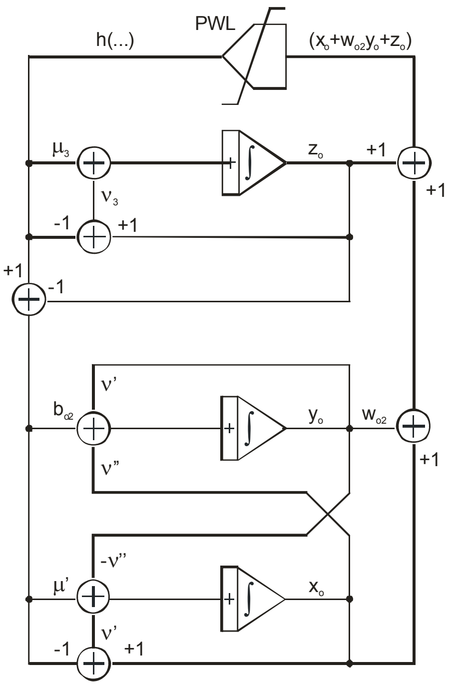

Fig. 2: Integrator-based circuit structure of the 3rd order state models with block-triangular state matrices and optimized 2nd order submatrix.

4 Relation between both forms

Except for the simple reference state model [4] having unity form of the partial transformation matrix $\tilde{\mathbf{K}}$ it is just the first canonical model that is suitable as qualitatively equivalent model for any other system of Class C . The partial transformation matrix of this state model has, in accordance with formula (8a), the simple form

$$\tilde{\mathbf{K}} = \mathbf{K_I} = \begin{bmatrix} 1 & 0 & 0 \\ q_1 & -1 & 0 \\ q_1^2 - q_2 & -q_1 & 1 \end{bmatrix} \tag{15}$$

and its inverse form is

$$\tilde{\mathbf{K}}^{-1} = \mathbf{K_I}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ q_1 & -1 & 0 \\ q_2 & -q_1 & 1 \end{bmatrix} . \tag{16}$$

The first canonical model is suitable as the reference form especially in the case, when the circuit model of some other dynamical system is utilized as prototype for practical realization of chaotic oscillators by the corresponding electronic circuit.

As shown in the Chapter 3 the state model optimized from the minimum of eigenvalue sensitivities represents the typical dynamical system of these properties. Substituting the definition formulas (11a) to (11d) into (8b), the corresponding partial transformation matrix $\mathbf{K} = \mathbf{K_{opt}}$ of this model has the form

$$\mathbf{K} = \mathbf{K_{opt}} = \begin{bmatrix} 1 & w_{o2} & 1 \\ \nu' + w_{o2}\nu'' & w_{o2}\nu' - \nu'' & \nu_3 - 2(\mu' - \nu') \\ K_{31} & K_{32} & K_{33} \end{bmatrix} , \tag{17}$$

where
$$K_{31} = (\nu')^2 - (\nu'')^2 + 2\nu'\nu'' w_{o2} ,$$
$$K_{32} = w_{o2} \lfloor (\nu')^2 - (\nu'')^2 \rfloor - 2\nu'\nu'' ,$$
$$K_{33} = \nu'' \lfloor b_{o2} - K(\nu'' - \mu'') \rfloor - 2(\mu' - \nu')(\nu' + \nu_3) .$$

The parameters b_{o2}, w_{o2} are given by formulas (11d) while the optimizing coefficient K by eqn (13).

The complete transformation matrix $\mathbf{T}$ can be expressed, in accordance with general formula (7d), as

$$\mathbf{T} = \mathbf{K_I}^{-1} \mathbf{K_{opt}} = \\ = \begin{bmatrix} 1 & w_{o2} & 1 \\ \nu' + \nu_3 - w_{o2}\nu'' & w_{o2}(\nu' + \nu_3) + \nu'' & 2\mu' \\ \nu_3(\nu' - w_{o2}\nu'') & \nu_3(w_{o2}\nu' + \nu'') & T_{33} \end{bmatrix} , \tag{18}$$

where $T_{33} = \nu'' \lfloor b_{o2} + \nu'' - K(\nu'' - \mu'') \rfloor + \nu'(2\mu' - \nu')$

and the parameters b_{o2}, w_{o2} are given by the same formulas (11d) as well as the optimizing coefficient K by eqn (13). Substituting the transformation matrix (18) to the basic linear topological conjugacy conditions (7a,b,c) the first canonical state model equivalent to its optimized form can be expressed.

5 Conclusion

Recently published state models of piecewise-linear (PWL) dynamical systems of Class C are suitable either for their numerical simulation (the first canonical form) or as prototypes for their circuit realization (the eigenvalue sensitivity optimized form – see Fig. 2). Their mutual relation is given analytically as linear topological conjugacy conditions with corresponding transformation matrix expressed by eqn (18) and graphically illustrated using the so-called "*elastic space analogy*" [4] – see Fig. 3.

Acknowledgement

This work has been supported by the Grant Agency of the Czech Republic, Grant projects No. 102/04/0469 and 102/05/0771 and by the Czech Ministry of Education, Research Programme No. MSM 0021630513, Advanced Electronic Communication Systems and Technologies.

References

[1] CHUA, L.O., "Global Unfolding of Chua's Circuit" *IEICE Trans. Fundamentals* E76-A, Vol. 5, 1993, pp.704-734.

[2] WU, C.W. and CHUA, L.O., "On Linear Topological Conjugacy of Lur'e Systems" *IEEE Trans. Circ. Syst. - I: Fundamentals...*, 43(2), 1996, pp. 158-161.

[3] POSPÍŠIL, J., BRZOBOHATÝ, J., "Elementary Canonical State Models of Chua´s Circuit Family" *IEEE Trans. CAS* 43, 1996, pp. 702-705.

[4] POSPÍŠIL, J., BRZOBOHATÝ, J., KOLKA, Z., and HORSKÁ, J., "Simplest ODE Equivalents of Chua's Equations", *Intern. Journ. of Bifurcation & Chaos*, 10(1), 2000, pp. 1-23 (Tutorial & Review paper).

[5] KOLKA, Z., "Synthesis of Optimized Piecewise-Linear System Using Similarity Transformation–Part I: Basic Principles" *Intern. Journal "Radio-engineering"*, vol. 10 (2001), no. 3, pp. 5-7.

[6] POSPÍŠIL, J., KOLKA, Z., HANUS, S. and BRZOBOHATÝ, J. "Synthesis of Optimized Piecewise-Linear System Using Similarity Transformation – Part III: Higher-Order Systems" *Intern. Journal "Radioengineering"*, vol. 11 (2002), no. 1, pp. 4-6.

[7] POSPÍŠIL, J, KOLKA, Z., HANUS, S., MICHÁLEK, V. and BRZOBOHATÝ, J. "Optimized State Model of PWL Dynamical Systems" *Intern. Journal "Radioengineering"*, vol. 12 (2003), no. 1, pp. 27-29.

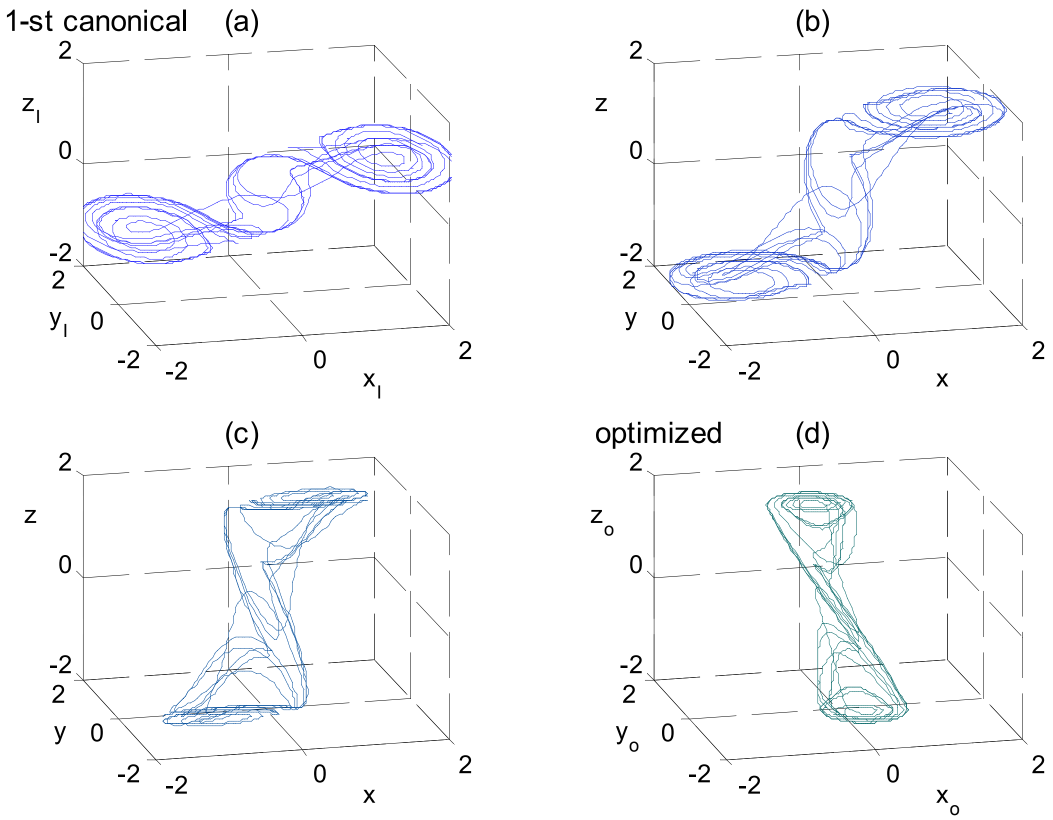

Fig. 3. Graphical illustration of linear topological conjugacy between the first canonical form and sensitivity optimized state model using the "*elastic space*" analogy. (a) First canonical form, (b)-(c) Transient steps, (d) Optimized state model. (eigenvalues: $\nu_{1,2} = 0.0610 \pm 1;$ $\nu_3 = -1.29;$ $\mu_{1,2} = -0.3190 \pm 0.8920;$ $\mu_3 = 0.7280$)