Analysis of the Anticipative Cobweb Model Dynamics

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ABSTRACT

The cobweb model of competitive market dynamics has been examined in the form of system dynamics model. The classical formulation of the cobweb demand-supply model has been transformed in the set of equations suitable for system dynamics modelling. The model has been expanded by the time parameter Δt which enables the stability analysis of the model for the change in the Δt. The structure of the model has been presented and key relationships in the model have been examined. The cyclical response of the system which is dependent on the demand$\sim$supply parameters and eigenvalues of the characteristic equation has been numerically examined.

Keywords: Cobweb, Anticipation, System Dynamics, Periodicity, Cycle

1. INTRODUCTION

Cobweb model presents the market demand-supply adjustment. It is typically viewed as the model of the agricultural pricing mechanism. The story behind the model is briefly explained as the common agriculture market adjustment mechanism: "The quantity offered for sale this year depends on what was planted at the start of the growing season, which in turn depends on the last year's price. Consumers look at the current prices, though, when deciding to buy. The cobweb model also assumes that the market is perfectly competitive and that supply and demand are both linear schedules." For clear and extensive introduction to the topic one should see [9]. In fact, the model derives from Evans model which discusses price $P(t)$ as a function of demand $D(t)$ and supplies $S(t)$ in the free market analyzed in previous research [6]:

$$P(t+\Delta t) \quad = \quad P(t)+\Delta t(D(t)-S(t)) \qquad (1)$$

For $\Delta t = 1$ we obtain the difference equation known as the cobweb Model. We will consider the demand and supply function under three different conditions:

$$D(t) \quad = \quad a_0 - a_1 P(t), S(t) = b_0 + b_1 P(t) \qquad (2)$$
$$D(t) \quad = \quad a_0 - a_1 P(t), S(t) = b_0 + b_1 P(t-1) \qquad (3)$$
$$D(t) \quad = \quad a_0 - a_1 P(t), S(t) = b_0 + b_1 P(t+1) \qquad (4)$$

The definition for parameters $a_0, a_1, b_0, b_1 > 0$ is always satisfied. For condition stated by Eq.2 assume a simultaneous time action for demand and suppliers. When $a_1 > b_1$, the price in Eq.1 is for any initial condition oscillatory transit to equilibrium, when $a_1 = b_1$ we have a stable cycle, and when $a_1 < b_1$ the systems are unstable. For condition stated by Eq.3 the suppliers' action is delayed when demand findings are the same as in case stated by Eq.2, where $a_1 > b_1$, and when $a_1 = b_1$ only the period and amplitude of the price is higher. When $a_1 < b_1$, the system is unstable but the amplitude is higher than in condition stated by Eq.2. In situation stated by Eq.4 suppliers anticipate the price on the market. In this case there is no price oscillation for the initial condition even when $a_1 < b_1$.

The results obtained in the simulation of price dynamics for the function of demand and supply with different information defined in situations determined by Eq.2, Eq.3 and Eq.4 show that information, which represents the rational expectation of price in situations determined by Eq.2 and Eq.3, does not remarkably influence the dynamics of price, though in situation determined by Eq.2 information regarding the price of goods is available immediately. Only the elongation about the equilibrium price is smaller. In case of Eq.4 the anticipation of price presenting a direct transition to the equilibrium from the initial value is observed. Eq.4 directly proves the possibility of rational expectation caused by anticipatory information. The modern communication means and better modelling (i.e. form of knowledge) could provide anticipatory information about stock, which represents the rational expectation of stock and can improve market behavior [10], [11], [12]. To speculate further: Some people probably can predict the market price by intuition and achieve a better position in the market. The average market price of goods is probably the average value of the portion of the situations determined by Eq.2, Eq.3 and Eq.4 situations governed by the actors.

The model in question therefore has all the characteristics of classical System Dynamics (SD) models: equilibrium, competitiveness, human perception, delay and adjustment but somehow it is avoiding to be settled in the common SD model bank of each SD modeler. The main reason for elusiveness of the cobweb model is in it's original form which is not suitable for the straight transformation to the common elements such as Level L and Rate R. The functions of demand $Q_d(k)$ and supply $Q_s(k)$ can be specified in the form:

$$\begin{aligned} Q_d(k) &= a + bP(k) &(5)\\ Q_s(k) &= c + dP(k-1) &(6)\end{aligned}$$

where a, b, c and d are parameters specific to individual markets. $P(k)$ and $Q_s(k)$ should be restricted to the positive values. In the cobweb model it is assumed that in any one time period producers supply a given amount (determined by the previous time period's price) and then price adjusts so that all the product supplied are bought by customers. If we write this in the form of equation then $Q_d(k) = Q_s(k)$ which enables us to state that the price is:

$$P(k) = \frac{d}{b}P(k-1) + \frac{c-a}{b} \quad (7)$$

Equations 5, 6 and 7 are not quite in the proper form in order to perform the transformation to the SD model. One of the things is the time argument $k-1$. The other is the missing Rate R elements and corresponding Δt. One should expect that the transformation will provide the known equations in the familiar cobweb model structure form. The developed model should enable us to examine the properties of the cobweb model and also to consider it's structural and incursive perspective. There are several approaches in modification and analysis of cobweb dynamics [3], [4], [5], [7]

2. SEPARATION OF SYSTEM ELEMENTS

Dynamically stable response indicates the periodical solution which will be of interest in further examination of the model. In general a solution y_n is periodic if $y_{n+m} = y_n$ for some fixed integer m and all n. The smallest integer for m is called period of the solution. In the classical cobweb model case the solution for dynamically stable system is a two-cycle solution. In general the following definition will be applied [8]:

Definition 2.1 *If a sequence $\{y_t\}$ has e.g. two repeating values y_1 and y_2, then y_1 and y_2 are called period points, and set $\{y_1, y_2\}$ is called a periodic orbit.*

Periodical response of the system is important because real agricultural systems depend on the cyclic behavior and could be controlled only by regarding the period of such systems. Examples from real cases could easily be found in crops as well as in the stock.

The classical structure (or more concise; notion) of the cobweb model states that the Price and Quantity are related. However the structure can be represented in a different way. By transforming the cobweb model to SD form the model could become non-autonomous depending on the variable Δt. The following two equations represent the different formulation of the cobweb model:

$$\begin{aligned} Q_s(k+1) &= c + d\frac{Q_s(k)-a}{b} &(8)\\ P(k+1) &= \frac{c + dP(k) - a}{b} &(9)\end{aligned}$$

This reformulation represents Q_s and P as the non-related quantities. The only bound that exists are the coefficients. In order to formulate the complete SD model the rate elements should be determined:

$$\begin{aligned} R_P(k) &= P(k+2) - P(k+1) = \\ &= \frac{d}{b}(P(k+1) - P(k)) &(10)\end{aligned}$$

$$\begin{aligned} R_{Q_s}(k) &= Q_s(k+2) - Q_s(k+1) = \\ &= \frac{d}{b}(Q_s(k+1) - Q_s(k)) &(11)\end{aligned}$$

In order to meet initial conditions of the model the $Q_s(k-1)$ should be determined:

$$Q_s(k-1) = a + \frac{b}{d}(Q(k) - c) \quad (12)$$

Equations for P and Q_s in standard SD form are the following:

$$\begin{aligned} P(k+1) &= P(k) + \Delta t R_P(k) &(13)\\ R_P(k) &= \frac{d}{b}\left(P(k) - \frac{bP(k) - c + a}{d}\right) &(14)\\ Q_s(k+1) &= Q_s(k) + \Delta t R_{Q_s}(k) &(15)\\ R_{Q_s}(k) &= \frac{d}{b}\left(Q_s(k) - \left(a + \frac{b}{d}(Q_s(k) - c)\right)\right) &(16)\end{aligned}$$

Eqs. 13, 14, 15 and 16 represents the cobweb model in separated form suitable for SD modelling. Note, that the terms for P and Q_s are related only to the coefficients a, b, c, d and p. $P(k+1)$ is dependent only on the value of $P(k)$ and coefficients a, b, c, d, p and not on the Q_s. Respectively for the $Q_s(k+1)$.

3. ANTICIPATIVE FORMULATION

The P and Q_s depend only on the parameter values a, b, c, d and p i.e. the initial conditions. Eqs. 13, 14, 15 and 16 enable the determination of entire anticipative (future event) chain while equation:

$$P(k-1) = \frac{bP(k) - c + a}{d} \tag{17}$$

and Eq.12 enable the determination of feedback (past event) chain. The dynamics of interest is therefore the chains dynamics which is dependant on the parameters $a(t)$, $b(t)$, $c(t)$, $d(t)$ and $p(t)$. Both chains are actually dependant on strategy dynamics which could be formulated as the $f(a, b, c, d, p, t)$.

Application of anticipative algorithm and inspection of gained equations with Dubois' [1] formulation of logistic growth and previous research [6] yields the following set of equations for the anticipative cobweb model:

$$\begin{aligned} P(k+2) &= \frac{d}{b}\Bigg(A - \Big(\frac{bB - c + a}{d}\Big)\Bigg) & (18) \\ Q_s(k+2) &= \frac{d}{b}\Bigg(C - a - \frac{b}{d}\Big(D - c\Big)\Bigg) & (19) \end{aligned}$$

with initial conditions:

$$\begin{aligned} P_0(k+1) &= \frac{p-a}{b} & (20) \\ P_0(k) &= \frac{bP_0(k+1) + a - c}{d} & (21) \\ Q_{s0}(k+1) &= p & (22) \\ Q_{s0}(k) &= a + \frac{b}{d}\Bigg(Q_{s0}(k+1) - c\Bigg) & (23) \end{aligned}$$

The coefficients A and B in Eq.18 could be replaced by the terms $P(k+1)$ or $P(k)$ while coefficients C and D in Eq.19 by $Q_s(k+1)$ or $Q_s(k)$. This yields 16 different combinations of system defined by Eq.18 and Eq.19 that should be studied.

The system combination further examined will have the following terms: $A = P(k+1), B = P(k), C = Q_s(k+1)$ and $D = Q_s(k)$. This yields the following set of equations:

$$\begin{aligned} P(k+2) &= \frac{d}{b}\Bigg(P(k+1) - \Big(\frac{bP(k) - c + a}{d}\Big)\Bigg) & (24) \\ Q_s(k+2) &= \frac{d}{b}\Bigg(Q_s(k+1) - a - \frac{b}{d}\Big(Q_s(k) - c\Big)\Bigg) & (25) \end{aligned}$$

Eq.24 and Eq.25 could be reformulated in order to show the dependency of the future-present-past events:

$$\begin{aligned} P(k) &= \frac{bP(k-1) + a - c}{d} + \frac{b}{d}P(k+1) & (26) \\ Q_s(k) &= \frac{b}{d}Q_s(k+1) + \frac{b}{d}Q_s(k-1) + a - \frac{bc}{d} & (27) \end{aligned}$$

Eq.26 and Eq.27 state that the value of the present is dependent on the past as well as on the future. This paradoxical statement is realizable since the formulation of feedback$\sim$anticipative chain could be stated. One might notice, that the level and rate elements are dependant only on the coefficients and initialization values.

4. RESULTS

Fig. 1 represents the helix-like response of the system at the parameters: $a = -12.43$, $b = 20$, $c = 18$, $d = -12.43$, $p = 160$, and $k = 4000$. The helix-like shape shows the behavior of the system which is bounded by the circular shape. The helix is transformed from e.g. pentagon helix to quad helix etc. There is an interesting dynamics that could be observed at the transitions from one cyclic equilibrium to another.

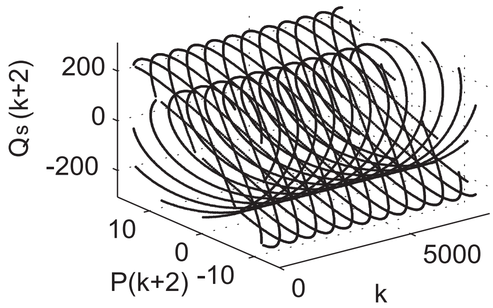

Figure 1: Helix response of the system.

The periodic responses of the system were gained according to the parameter values gathered in Table 1. The change was made in parameter d which yielded the synchronization patterns as shown by the shape column. The parameter values were gained by the simulation where the the range of parameter d was set as of $[-40,\ 40]$ with $\Delta d = 0.001$. The condition for parameter values determination was set by the rule of acceptable error between simulation steps and definition 2.1 of synchronization .

Equilibrium condition for the P segment of the hyperincursive cobweb system could be stated as:

$$\frac{d}{b}\Bigg(\frac{p-a}{b} - \frac{\frac{p-c}{d} - c + a}{d}\Bigg) = \frac{p-a}{b} = \frac{p-c}{d} \tag{28}$$

The equilibrium values of the parameters for the P segment of the system are: $a = c = p$ and $b = d$.

Equilibrium condition for the Q_s segment of the hyper-incursive cobweb model could be stated as:

$$\frac{d}{b}\left(p - a - \frac{b}{d}\left(a + \frac{b}{d}(p-c) - c\right)\right) = a + \frac{b}{d}(p-c) \quad (29)$$

Q_s segment of the system has no solution for the equilibrium values of the parameters. When the equilibrium conditions for the P segment of the system are considered than in fact in all the cases the Q_s segment of the system could not be in stable state. Graphical presentation of the equilibrium conditions $a = c = p$ and $b = d$ is shown in Fig. 2.

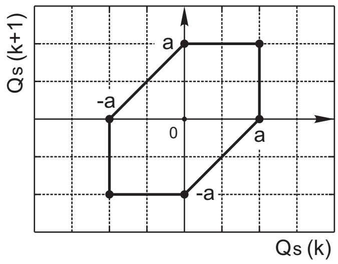

Figure 2: Response of the Q_s segment of the system while in equilibrium

Corollary 4.1 *Equilibrium condition for the P segment of the anticipative cobweb system defined by the equations from Eq.20 to Eq.25 is: $a = c = p$ and $b = d$. At this conditions, the Q_s segment of the system has the response of hexagon shape with vertices $\{(a, 0), (a, a), (0, a), (-a, 0), (-a, -a), (0, -a)\}$ in $Q_s(k)$, $Q_s(k + 1)$ mapping.*

While the response of the system for the P segment is in equilibrium, the Q_s segment of the system has the hexagon-like shape with significant dimension of parameter a value and edges dimension of a and $a\sqrt{2}$.

Table 1: Synchronization parameter values

Shape	Description	a	b	c	d	p
△	$Triangular$	400	-20	-50	20.0000	160
◇	$Quad$	400	-20	-50	-0.0010	160
⬠	$Pentagon$	400	-20	-50	-12.3671	160
☆	$Pentagram$	400	-20	-50	32.3620	160
⬡	$Hexagon$	400	-20	-50	-20.0000	160
✪	$Nonagram$	400	-20	-50	-6.9450	160
✺	$Hexagram$	400	-20	-50	15.3070	160

Fig. 3 shows the change in direction coefficients k_P and k_{Qs} for the demand/supply curves. The intersection of the curves and poles indicates the periodicity response of the system. The range of the cyclical behavior is determined by the classical imaginary solution of the dynamical system which is in our case defined by the condition

$$\lambda = \frac{-2b + d \pm \sqrt{-4b^2 + d^2}}{2b} \quad (30)$$

Corollary 4.2 *Triangular (△) i.e. three-period response in 2-dimensional mapping is determined by the condition $b = -d$.*

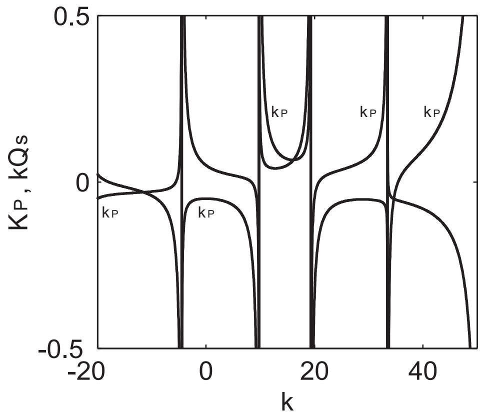

Figure 3: Direction coefficients for the cobweb response of the system

In order to gain the term for the period in the P values one should apply Def.2.1. The values for times $1, \ldots, 4$ should be symbolically expressed. By inserting the Eq.20 and Eq.21 into Eq.24 the following term is gained:

$$P(k+2) = \frac{d}{b}\left(\frac{p-a}{b} - \frac{a - c + \frac{b(p-c)}{d}}{d}\right) \quad (31)$$

By repetition of similar procedure the equation for $P(k+3)$ considering the period 4 condition i.e.: $P(k+3) = P(k)$ one should get the following equation:

$$\frac{d}{b}\left(\frac{d}{b}\left(\frac{p-a}{b} - \frac{a - c + \frac{b(p-c)}{d}}{d}\right) - \frac{p-c}{d}\right) - \frac{p-c}{d} = 0$$

with solution $b = -d \in \Re/\{0\}$.

Fig. 4 shows parametric graph of the change coefficients k_P and k_{Qs}. The sequence of parametric plot is marked with the numbers next to the graph lines.

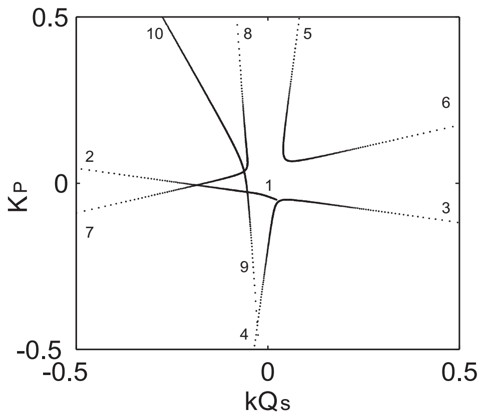

Figure 4: Sequence on the parametric graph of direction coefficients k_P and k_{Qs}

5. CONCLUSION

The story revealed from the developed hyperincursive model rises the following questions: a) Does change in the strategy change the structure or does it change only the relations between the elements of the structure? b) Does changing of the strategy change the future as well as the past? Change in the strategy would mean new and different future and should also mean different past if the change in the strategy would occur earlier. The hyperincursive cobweb model enables us to change the future as well as the past chain of events. However different examination of the system dynamics is proposed where change in the key parameters is performed while observing the change in complete future and past chain rather than observing the classical time response of the system.

The following procedure proposition emerges which enabled the anticipative formulation of the classical dynamic system: since the hyperincursive systems are hard to determine [2], [1], the developed anticipatory mechanisms should be applied therefore the model should be a) transformed in the separated form b) provide past-future chain property and c) apply the hiperincursive structure to the studied model.

The developed model shows that by the statement of general rule of the system the synchronization of entire feedback-anticipative chain could be gained by setting the appropriate strategy in the form of parameters value set which should be time dependant.

The idea for the simulation proposed in the paper is quite different from the common paradigm. The structure of the model should yield the entire feedback-anticipative chain and the observation of the entire system response should be made. This provides the new and quite challenging responses which should initiate further interest and examination of proposed model.

One of the interesting responses from the model is the helix like shape which is synchronized at the certain time steps. Entire feedback-anticipative chain i.e. all point set is synchronized according to the period of the system.

6. *ACKNOWLEDGEMENT*

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