Reducing Index Method for Differential Algebraic Equations

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Abstract

In [Appl. Math. Comput. In press] a reducing index method has been proposed for some cases of semi-explicit DAEs(differential algebraic equations). In this paper, this method is generalized to more cases. Also, numerical implementation of generalized method is presented by pseudospectral method. In addition, aforementioned methods will be considered by one example.

Keywords- Differential-algebraic equations, Hessenberg systems, Index reduction techniques, pseudospectral method.
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1 Introduction

It is well known that differential algebraic equations can be difficult to solve when it has a higher index (index greater than one, [1]). Higher index DAEs are ill posed and an alternative treatment is the use of index reduction methods (see, e.g., [4,5,7,10]), whose essence is the repeated differentiation of the constraint equations until a low-index problem (an index-1 DAEs or ordinary differential equations) is obtained. But repeated index reduction by direct differentiation leads to instability of the resulting ODE, and this causes drift-off the numerical error in the original constraint grows. In this case, stabilized index reduction methods were used to overcome the difficulty. In [3,8,9], for some cases of linear semi explicit DAEs, with higher index, an efficient reducing index method was proposed which has not need to the repeated differentiation of the constraint equations. This method was well solved the DAEs with and without constraint singularities and them were numerically solved by pseudospectral method with and without domain decomposition. Here, this proposed reducing index method is extended to general case of linear semi explicit DAEs.

Now consider a linear (or linearized) semi-explicit DAEs:

$$X^{(m)} = \sum_{j=1}^{m} A_j X^{(j-1)} + By + q, \qquad (1a)$$
$$0 = CX + r, \qquad (1b)$$

where, A_j, B and C are smooth functions of t, $t_0 \leq t \leq t_f$, $A_j(t) \in R^{n\times n}, j = 1, ..., m, B(t) \in R^{n\times k}$, $C(t) \in R^{k\times n}, n > k$, $n \geq 2$, and CB is nonsingular (the DAEs has index $m+1$). The homogeneities are $q(t) \in R^n$ and $r(t) \in R^k$. Early in [3,8,9] , for $k = 1$, the index of problem (1) has been reduced by introducing a simple formulation. In this paper, we will reduce the index of (1) when $k > 1$. For this reason, we put

$$y = (CB)^{-1} C[X^{(m)} - \sum_{j=1}^{m} A_j X^{(j-1)} - q], \qquad (2)$$

and by substituting (2) into (1.a), we obtain an implicit DAE which has index m, as follows,

$$\sum_{j=0}^{m} E_j X^{(j)} = \widehat{q}, \qquad (3)$$

where $E_j(t) \in R^{n\times n}, j = 0, 1, ..., m$, and except $E_0(t)$, others are singular matrices. Note that system (3) has k equations less than system (1).

2 A simple formulation for index reduction

In this section, DAEs (1) is considered when $m = 1$ and $k = 2$. To extend it to general case (1) is easy. Now consider the Hessenberg index-2 system,

$$X' = AX + By + q, \qquad (4a)$$
$$0 = CX + r. \qquad (4b)$$

where, $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times 2}$, $C = (c_{ij})_{2\times n}$, $n \geq 3$ and

$$\det(CB(t)) \neq 0 \qquad t \in [t_0, t_f] \qquad (5)$$

From (4a) and (5), we can write

$$y = (CB)^{-1}C[X' - AX - q], \qquad t \in [t_0, t_f] \quad (6)$$

and substituting (6) into (4a), implies,

$$X' = AX + B(CB)^{-1}C[X' - AX - q] + q.$$

So, problem (4) transforms to the system:

$$\det(CB(t))[I - B(CB)^{-1}C][X' - AX - q] = 0, \qquad (7a)$$
$$CX + r = 0, \qquad (7b)$$

Here, the overdetermined system (7) will be transformed to a full rank DAE system with n equation and n unknowns which has index one.

Theorem 1. The index-2 DAE system (4), with n=3, is equivalent to index-1 DAE system (8),

$$\begin{bmatrix} \overline{M} \\ 0 \end{bmatrix} X' + \begin{bmatrix} -\overline{M}A \\ C \end{bmatrix} X = \begin{bmatrix} \overline{M}q \\ -r \end{bmatrix}, \qquad (8)$$

such that,

$$\overline{M} = \left[\begin{array}{ccc} b_{21}b_{32} - b_{22}b_{31} & b_{12}b_{31} - b_{11}b_{32} & b_{11}b_{22} - b_{12}b_{21} \end{array}\right], \quad (9)$$

Proof : As it is seen, the DAE system (4) is transformed to overdetermined system (7) by using (6). Since $n = 3$, we have,

$$\det(CB(t)) = (c_{11}c_{22} - c_{12}c_{21})(b_{11}b_{22} - b_{12}b_{21}) + (c_{11}c_{23} - c_{13}c_{21})(b_{11}b_{32} - b_{12}b_{31}) + (c_{12}c_{23} - c_{13}c_{22})(b_{21}b_{32} - b_{22}b_{31}),$$

Now, if we define

$$c_1 = c_{11}c_{22} - c_{12}c_{21}, \quad c_2 = c_{11}c_{23} - c_{13}c_{21}, \quad c_3 = c_{12}c_{23} - c_{13}c_{22},$$
$$b_1 = b_{11}b_{22} - b_{12}b_{21}, \quad b_2 = b_{11}b_{32} - b_{12}b_{31}, \quad b_3 = b_{21}b_{32} - b_{22}b_{31},$$

then we can rewrite $\det(CB(t))$ as below,

$$\det(CB(t)) = (c_1b_1 + c_2b_2 + c_3b_3)(t) \neq 0. \qquad t \in [t_0, t_f] \qquad (10)$$

Also, we have

$$M_{n\times n} = \det(CB(t))\,[I - B(CB)^{-1}C] =$$

$$\left[\begin{array}{ccc} c_3b_3 & -c_3b_2 & c_3b_1 \\ -c_2b_3 & c_2b_2 & -c_2b_1 \\ c_1b_3 & -c_1b_2 & c_1b_1 \end{array}\right], \qquad (11)$$

(10) and (11) imply that $rank(M) = 1$. In addition, if we define $\overline{M} = \left[\begin{array}{ccc} b_3 & -b_2 & b_1 \end{array}\right]$, by considering (7), we have

$$\overline{M}\left[X' - AX - q\right] = 0,$$
$$CX + r = 0,$$

and it implies that

$$\left[\begin{array}{c} \overline{M} \\ 0 \end{array}\right] X' + \left[\begin{array}{c} -\overline{M}A \\ C \end{array}\right] X = \left[\begin{array}{c} \overline{M}q \\ -r \end{array}\right],$$

So, the overdetermined system (7) is transformed to system (8), with 3 equations and unknowns. In continuation, we must show that the system (8) is full rank and has index 1. For this reason, it is sufficient to show that

$\begin{bmatrix} \overline{M} \\ C \end{bmatrix}_{n\times n}$ is nonsingular (according to algorithm (4.1) mentioned in [7]). But by computing the determinant of $\begin{bmatrix} \overline{M} \\ C \end{bmatrix}_{n\times n}$, we have

$$\det\left(\begin{bmatrix} \overline{M} \\ C \end{bmatrix}\right) = \det\left(CB(t)\right) \neq 0. \qquad t \in [t_0, t_f]$$

Hence, by theorem 1 a simple formulation is presented to reduce the index of DAE system (4) when $n = 3$. For $n > 3$, to present a simple formulation, as well as $n = 3$, is not possible and to reduce the index of (4), it is need to impose a condition as below.

Theorem 2: Consider the index-2 DAEs system (4) with n=4, the $i-th$ and $j-th$ rows of $M = \det(CB(t))\left[I - B(CB)^{-1}C\right]$ are linearly dependent to other rows of M if

$$\left(c_{1i}c_{2j} - c_{1j}c_{2i}\right)(t) \neq 0 \qquad t \in [t_0, t_f] \qquad (12)$$

Proof : Through using a Maple program, we conclude that

$$\det(CB(t)) = \sum_{i=1}^{4-1}\sum_{j=i+1}^{4}\left(c_{1i}c_{2j} - c_{1j}c_{2i}\right)\left(b_{i1}b_{j2} - b_{j1}b_{i2}\right), \qquad (13)$$

and if $\overline{M}$ is obtained by eliminating the $i-th$ and $j-th$ rows of M, then we have $rank(M) = rank(\overline{M}) = 2$.

In addition we have

$$\det\left(\begin{bmatrix} \overline{M} \\ C \end{bmatrix}\right) = \left(c_{1i}c_{2j} - c_{1j}c_{2i}\right) \times \det{}^2(CB),$$

which it implies that

$$rank\left(\begin{bmatrix} \overline{M} \\ C \end{bmatrix}_{4\times 4}\right) = 4.$$

Hence, the obtained system,

$$\overline{M}\left[X' - AX - q\right] = 0,$$
$$CX + r = 0,$$

is full rank and has index 1. It must be noted that, since $\det(CB(t)) \neq 0$, hence according to (13), there exist i and j such that condition (12) is hold. Also the above discussion can be easily extended to DAEs problem with constraint singularities(according to [8]) and to reduce index of problem (4) with $k \geq 3$ we can apply the similar way by using an appropriate maple program.

3 Implementation of numerical method

Here, the implementation of pseudospectral method is presented for DAEs systems (4) and (8). This discussion can simply be extended to general forms (1) and (3). Now consider the DAEs systems,

$$\sum_{j=1}^{3} f_j(t)x_j' + \sum_{j=4}^{6} f_j(t)x_{j-3} = \widehat{q}(t), \qquad (14a)$$
$$\sum_{j=1}^{3} c_{ij}(t)x_j = -r_i(t), \qquad i = 1, 2 \qquad (14b)$$

with initial condition,

$$x_1(t_0) = \alpha, \qquad (15)$$

For an arbitrary natural number ν, we suppose that the approximate solution of DAEs systems (14) is as below,

$$x_j(t) = \sum_{i=0}^{\nu} a_{i+(j-1)\times(\nu+1)}T_i(s), \qquad j = 1, 2, 3 \qquad s \in [-1, 1] \qquad (16)$$

where

$$t = h(s) = \frac{t_f - t_0}{2}s + \frac{t_f + t_0}{2}, \qquad (17)$$

where $\mathbf{a} = (a_0, a_1, ..., a_{3\nu+2})^t \in R^{3\nu+3}$ and $\{T_k\}_{k=0}^{\infty}$ is sequence of Chebyshev polynomials of the first kind. Here, the main purpose is to find vector

a. Now, by using (17), we rewrite system (14) and (15), as below,

$$\sum_{j=1}^{3}\frac{ds}{dt}f_j(h(s))x_j' + \sum_{j=4}^{6}f_j(h(s))x_{j-3} = \widehat{q}(h(s)), \qquad s\in[-1,1] \quad (18a)$$
$$\sum_{j=1}^{3}c_{ij}(h(s))x_j = -r(h(s)), \qquad i=1,2 \quad (18b)$$
$$x_1(-1)=\alpha, \quad (18c)$$

By substituting (16) into (18) we have(for more details refer to [2,3])

$$\sum_{i=0}^{3\nu+2}a_i\Phi_i(s)\approx\widehat{q}(s), \quad (19a)$$
$$\sum_{i=0}^{3\nu+2}a_i\Psi_{ij}(s)\approx -r_j(s), \qquad j=1,2 \quad (19b)$$
$$\sum_{i=0}^{\nu}a_iT_i(-1)=\alpha \quad \Rightarrow \quad \sum_{i=0}^{\nu}a_i(-1)^i=\alpha. \quad (19c)$$

Relation (19c) forms a system with one equation and $3\nu+3$ unknowns, to construct the remaining $3\nu+2$ equations we substitute Chebyshev-Guass points,

$$s_j=\cos(\tfrac{2\pi j}{2\nu}) \qquad j=0,1,..,\nu-1$$

in (19a) and

$$s_j=\cos(\tfrac{2\pi j}{2\nu}) \qquad j=0,1,..,\nu$$

in (19b) to obtain $3\nu+2$ equations.

4 Numerical example

Here, we use "e_x" and "e_y" to denote the maximum absolute error in vectors $X=(x_1,x_2,x_3)$ and $y=(y_1,y_2)$. These values are approximately obtained through their graphs. Results show the advantages of techniques, mentioned in sections 2 and 3. Also, the presented algorithm in section 3, is performed by using Maple 8 with 25 digits precision.

Example 1. Consider for $0\le t\le 1$,

$$X'=AX+By+q, \quad (20a)$$

$$0 = CX + r, \qquad (20b)$$

where

$$A = \begin{bmatrix} \frac{3-2t}{2-t} & 0 & 0 \\ \frac{1}{2-t} & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 4-2t & 0 \\ 0 & 1 \\ \sin(2t) & \cos(2t) \end{bmatrix}, \quad C = B^T,$$

with initial condition, $x_1(0) = 1$,and exact solutions, $x_1(t) = x_2(t) = x_3(t) = e^t$,and $y_1(t) = y_2(t) = \frac{e^t}{t-2}$. $q(t)$ and $r(t)$ are compatible with above exact solutions. This problem has index 2.

$$\det(CB) = 1 + \sin^2(2t) + 4(2-t)^2 + 4(t-2)^2\cos^2(2t) \neq 0, \qquad t \in [0,1]$$

Also according to (9),

$$\overline{M} = \begin{bmatrix} -\sin(2t) & (2t-4)\cos(2t) & 4-2t \end{bmatrix}.$$

Hence, by theorem 1 the index-2 DAE (20) converts to index-1 DAE (21) as below,

$$\begin{bmatrix} \overline{M} \\ 0 \end{bmatrix} X' + \begin{bmatrix} -\overline{M}A \\ C \end{bmatrix} X = \begin{bmatrix} \overline{M}q \\ -r \end{bmatrix}, \qquad (21)$$

In table 1, we record the results of running pseudospectral method for example 1 with and without index reduction.

Maximum norm error for example 1

ν	Without index reduction		With index reduction	
	e_x	e_y	e_x	e_y
6	2.7(−3)	4.2(−3)	4.5(−4)	2.3(−4)
8	1.6(−4)	2.5(−4)	1.4(−6)	1.0(−6)
12	2.7(−6)	4.4(−6)	8.0(−12)	3.0(−12)
16	1.4(−7)	2.3(−7)	5.0(−17)	3.4(−17)

Table 1

The advantage of using index reduction method (proposed in sections 2) is clearly demonstrated for above example.

5 REFERENCES

1. U. M. Ascher and P. Lin, Sequential regularization methods for higher index differential-algebraic equations with constraint singularities: the linear index-2 case, SIAM J. Anal., 33: 1921-1940 (1996).

2. E. Babolian and M. M. Hosseini, A modified spectral method for numerical solution of ordinary differential equations with non-analytic solution, Applied Mathematics and Computation, 132 (2002) 341-351.

3. E. Babolian and M. M. Hosseini, Reducing index, and pseudospectral methods for differential-algebraic equations, Applied Mathematics and Computation, 140 (2003) 77-90.

4. K. E. Brenan, S.L. Campbell and L. R. Petzold, Numerical solution of initial-value problems in differential-algebraic equations, Elsevier, New York, 1989.

5. S. L. Campbell, A computational method for general higher index singular systems of differential equations, IMACS, Trans. Sci. Comput., 89 (1989) 555-560.

6. C. Canuto, M. Y. Hussaini, A. Quarteroni and A. Zang, Spectral methods in fluid dynamics, Springer-Verlag, 1998.

7. C. W. Gear and L. R. Petzold, ODE methods for the solution of differential/algebraic systems, SIAM J. on Numerical Analysis, 21:716-728, (1984).

8. M. M. Hosseini, Reducing index method for differential-algebraic equations with constraint singularities, Applied Mathematics and Computations, 153 (2004) 205-214.

9. M. M. Hosseini, Numerical solution of linear differential-algebraic equations, J. Applied Mathematics and Computation, In Press.

10. H. Wang and Y. Song, Regularization methods for solving differential-algebraic equations, Applied Mathematics and Computation, 119 (2001) 283-296.