Piezewise polynomial approximations for weakly singular integral equations with discontinuous coefficients

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Abstract: We study the attainable order of a piecewise polynomial collocation method for the numerical solution of linear integral equations with weakly singular or other nonsmooth kernels. In particular, the kernel may have the form \( K(t, s) = g(t, s)|t - s|^{-\nu}, \) \( 0 < \nu < 1, \) where \( g \) is proposed to be smooth only on \( [0, b] \times ([0, b] \setminus \{d\}), 0 < d < b. \) We show that the proposed method is of maximal possible order if the grid is chosen appropriately.

Key words: Fredholm integral equation, weakly singular kernel, collocation method.

1 Introduction

Let \( \mathbb{R} = (-\infty, \infty), \mathbb{N} = \{1, 2, \ldots\}, \mathbb{N}_0 = \{0\} \cup \mathbb{N}. \) For \( \Omega \subset \mathbb{R}^n, \) by \( C^m(\Omega) \) we denote the set of \( m \) times continuously differentiable functions \( x: \Omega \to \mathbb{R}, C^0(\Omega) = C(\Omega). \) The set \( C[a, b] \) of continuous functions \( x: [a, b] \to \mathbb{R} \) is a Banach space with respect to the norm \( \|x\|_{C[a, b]} = \max_{a \leq t \leq b} |x(t)|. \)

Let us consider an integral equation of the form

\[
u(t) - \int_0^b K(t, s)\nu(s)ds = f(t), \quad 0 \leq t \leq b,
\]

with \( f \in C[0, b] \) and \( K(t, s) = g(t, s)|t - s|^{-\nu}, \) \( 0 < \nu < 1, \) where \( g \) is a sufficiently smooth function on \( [0, b] \times [0, b]. \) Solutions of integral equations of this type will in general contain singularities in their derivatives at the endpoints of the interval \( [0, b], \) even for smooth forcing functions \( f \) (see, for example, \( [1, 5, 6]). \) Therefore difficulties in constructing of high order numerical methods for solving (1) arise. To overcome these difficulties, one can thicken near 0 and \( b, \) the grid which is used to built approximate solution \( \{1, 5, 6\}. \)

In the present paper we study the case if \( g \) is proposed to be smooth only on \( [0, b] \times ([0, b] \setminus \{d\}), \) with \( d \in (0, b). \) In this case the derivatives of the solution \( \nu(s) \) of equation (1) may have singularities at \( s = d, \) also \( \{4, 5\}. \) Therefore, to get numerical algorithms of higher order for solving (1), we shall thicken the grid near \( s = d, \) too. In fact, we shall construct a piecewise polynomial collocation method for the numerical solution of a wide class of weakly singular integral equations and show that it is of maximal possible order if the grid is chosen appropriately.

2 Smoothness of the solution

We consider a kernel \( K \) in the form

\[
u(t, s) = g(t, s)\kappa(t, s)
\]

with \( g \) and \( \kappa \) satisfying the following assumptions (A1) and (A2), respectively.

(A1) The function \( g = g(t, s) \) is \( m \) times \( (m \geq 1) \) continuously differentiable with respect to \( t \) and \( s \) for \( t \in [0, b], s \in [0, b] \setminus \{d\}, 0 < d < b, \) and its derivatives are bounded in the regions \( [0, b] \times [0, d] \) and \( [0, b] \times (d, b]. \) Let \( p \) \( (0 \leq p \leq m) \) be an integer defined as follows: \( p = 0 \) if \( g \) may have a discontinuity across the line \( s = d; p \geq 1 \) if \( g \in C^{p-1}([0, b] \times [0, b]). \)

(A2) The function \( \kappa = \kappa(t, s) \) is \( m \) times \( (m \) is fixed in the assumption (A1)) continuously differentiable with respect to \( t \) and \( s \) for \( t, s \in [0, b], t \neq s, \) and there exists a real
number \( \nu, -\infty < \nu < 1 \), such that the estimate
\[
\left| \left( \frac{\partial}{\partial t} \right) \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right)^j \kappa(t, s) \right| \leq c \left\{ \begin{array}{ll}
1, & \text{if } \nu + i < 0, \\
1 + |\ln |t - s||, & \text{if } \nu + i = 0, \\
|t - s|^{-\nu - i}, & \text{if } \nu + i > 0,
\end{array} \right.
\]
holds with a positive constant \( c \) for all \( t, s \in [0, b], t \neq s \) and for all \( i, j \in \mathbb{N}_0, i + j \leq m \).

For \( i = j = 0 \), condition (3) yields
\[
|\kappa(t, s)| \leq c \left\{ \begin{array}{ll}
1, & \text{if } \nu < 0, \\
1 + |\ln |t - s||, & \text{if } \nu = 0, \\
|t - s|^{-\nu}, & \text{if } \nu > 0.
\end{array} \right.
\]
Thus, a kernel (2) is at most weakly singular for \( 0 \leq \nu < 1 \). For \( \nu < 0 \), the kernel (2) is bounded but its derivatives may have diagonal singularities. Most important examples of kernels of type (2) are given by
\[
K(t, s) = g(t, s)|t - s|^{-\nu}, \quad 0 < \nu < 1,
K(t, s) = g(t, s) \ln |t - s|,
\]
where \( g \) is a function which satisfies the condition (A1).

For equations (1) with smooth kernels, the smoothness of the kernel \( K \) and the forcing function \( f \) determines the smoothness of the solution \( u \) on the closed interval \([0, b]\). If we allow weakly singular kernels of type (2), with smooth coefficient functions \( g: [0, b] \times [0, b] \to \mathbb{R} \), then the resulting solutions are typically nonsmooth at the endpoints of the interval of integration \([0, b]\), where their derivatives become unbounded. If \( g \) is proposed to be smooth only on \([0, b] \times ([0, b] \setminus \{d\})\), where \( 0 < d < b \), then the derivatives of the solution \( u(t) \) of equation (1) may have singularities at \( t = d \), also (see Lemma 1 below). In order to characterize those singularities we introduce a set of functions \( C^{m, \nu}_{d, p}[0, b] \).

Let \( m \in \mathbb{N}, \nu \in \mathbb{R}, \nu < 1, 0 < d < b, p \in \mathbb{N}_0, p \leq m \). Define \( C^{m, \nu}_{d, p}[0, b] \) as the collection of continuous functions \( u: [0, b] \to \mathbb{R} \) which are \( m \) times continuously differentiable in \((0, b) \setminus \{d\}\) and such that the estimate
\[
|u^{(j)}(t)| \leq \left\{ \begin{array}{ll}
1, & \text{if } j < 1 - \nu, p \in \{0, 1, \ldots, m\}; \\
1 + |\ln t| + |\ln (b - t)|, & \text{if } j = 1 - \nu, p \in \{1, \ldots, m\}; \\
1 + |\ln t| + |\ln |d - t|| + |\ln (b - t)|, & \text{if } j = 1 - \nu, p = 0; \\
t^{1 - \nu - j} + (b - t)^{1 - \nu - j}, & \text{if } 1 - \nu < j < 1 - \nu + p, p \in \{1, \ldots, m\}; \\
t^{1 - \nu - j} + |\ln |d - t|| + (b - t)^{1 - \nu - j}, & \text{if } j = 1 - \nu + p, p \in \{1, \ldots, m - 1\}; \\
t^{1 - \nu - j} + |d - t|^{1 - \nu - j + p} + (b - t)^{1 - \nu - j}, & \text{if } j > 1 - \nu + p, p \in \{0, \ldots, m - 1\},
\end{array} \right.
\]
holds with a positive constant \( c = c(u) \) for every \( t \in (0, b) \setminus \{d\} \) and \( j = 1, \ldots, m \).

The following result characterizes the regularity properties of solutions to equation (1), see [4,5].

**Lemma 1.** Let the conditions (A1) and (A2) about the kernel (2) be fulfilled. Let \( f \in C^{m, \nu}_{d, p}[0, b] \), with \( m, \nu, d, p \), fixed in the assumptions (A1) and (A2). If the integral equation (1) has an integrable solution \( u \in L^1(0, b) \) then \( u \in C^{m, \nu}_{d, p}[0, b] \).

### 3 Piecewise polynomial interpolation

For given \( N = 4n, n \in \mathbb{N}, b, d, r, r_d \in \mathbb{R}, 0 < d < b, r, r_d \geq 1 \), let
\[
\Delta_N = \{ t_0, t_1, \ldots, t_N : 0 = t_0 < t_1 < \ldots < t_N = b \}
\]
be a partition (a grid) for the interval \([0, b]\) with the following nodes \( t_0, \ldots, t_N \):
\[
t_j = \frac{d}{2} \left( \frac{j}{n} \right)^r, \quad j = 0, 1, \ldots, n;
\]
\[
t_{n+j} = d - \frac{d}{2} \left( \frac{n-j}{n} \right)^r, \quad j = 1, \ldots, n;
\]
\[
t_{2n+j} = d + \frac{b-d}{2} \left( \frac{j}{n} \right)^r, \quad j = 1, \ldots, n;
\]
\[
t_{3n+j} = b - \frac{b-d}{2} \left( \frac{n-j}{n} \right)^r, \quad j = 1, \ldots, n.
\]
Then \( \Delta_N \) is called a graded grid for \([0, b]\). In the present context the so-called grading exponents \( r, r_d \) will always satisfy \( r \geq 1 \) and \( r_d \geq 1 \). These parameters characterize the accumulation of nodes.
$t_0, t_1, \ldots, t_N$ near the points of possible unboundedness of the derivatives of the solution $u$ of equation (1) (see Lemma 1). For larger $r$ and $r_d$ the grid $\Delta_N$ is thicker near 0, d and b. We use two different parameters $r$ and $r_d$ because the order of singularity of the solution $u$ can be different at points 0, b and d. If $r = r_d = 1$ then the grid points (5) are uniformly located in the intervals $[0, d]$ and $[d, b]$.

It follows from (5) that an estimate

$$h_N \equiv \max_{j=1, \ldots, N} (t_j - t_{j-1}) \leq c N^{-1} \quad (6)$$

holds with a positive constant $c$ which is independent of $N$.

For $m \in \mathbb{N}_0$, let $S_m^{(0)}(\Delta_N)$ and $S_m^{(-1)}(\Delta_N)$ be the spline spaces of piecewise polynomial functions on the grid $\Delta_N$:

$$S_m^{(0)}(\Delta_N) = \{ u \in C[0, b] : u|_{\sigma_j} \in \pi_m, j = 1, \ldots, N \}, \quad S_m^{(-1)}(\Delta_N) = \{ u : u|_{\sigma_j} \in \pi_m, j = 1, \ldots, N \}. \quad (7)$$

In (7) $\pi_m$ denotes the set of polynomials of degree not exceeding $m$ and $u|_{\sigma_j}$ is the restriction of $u$ to the subinterval $\sigma_j = [t_{j-1}, t_j]$ ($j = 1, \ldots, N$). Note that the elements of $S_m^{(-1)}(\Delta_N)$ may have jump discontinuities at the interior grid points $t_1, \ldots, t_{N-1}$.

In every subinterval $[t_{j-1}, t_j]$, $j = 1, \ldots, N$ we define $m \in \mathbb{N}$ interpolation points

$$\xi_{j,q} = t_{j-1} + \frac{\eta_q + 1}{2}(t_j - t_{j-1}), \quad q = 1, \ldots, m; \quad j = 1, \ldots, N, \quad (8)$$

where

$$-1 \leq \eta_1 < \ldots < \eta_m \leq 1 \quad (9)$$

is some fixed system of $m$ parameters on the interval $[-1, 1]$, which is the same for every $j$ and $N$.

To a given continuous function $u : [0, b] \rightarrow \mathbb{R}$ we assign a piecewise polynomial interpolation function $P_N u = P_{N,m-1} u \in S_m^{(-1)}(\Delta_N)$ which interpolates $u$ at the nodes (8). Let $P_N = P_{N,m-1} : C[0, b] \rightarrow S_m^{(-1)}(\Delta_N)$ be an interpolation operator which assigns to every continuous function $u : [0, b] \rightarrow \mathbb{R}$ its piecewise interpolation function $P_N u:

$$P_N u \in S_m^{(-1)}(\Delta_N), \quad u \in C[0, b], \quad (P_N u)(\xi_{j,q}) = u(\xi_{j,q}), \quad q = 1, \ldots, m; \quad j = 1, \ldots, N. \quad (10)$$

Thus, $(P_N u)(t)$ is independently defined in every subinterval $[t_{j-1}, t_j]$ ($j = 1, \ldots, N$) and may be discontinuous at $t = t_j, j = 1, \ldots, N - 1$; we can treat $P_N u$ as a two-valued function at these points. If $\eta_1 = -1, \eta_m = 1$ then $P_N u$ is a continuous function on the interval $[0, b]$.

Let $E$ and $F$ be Banach spaces. By $\mathcal{L}(E, F)$ we denote the Banach space of all linear bounded operators $A : E \rightarrow F$ with the norm $\| A \|_{\mathcal{L}(E, F)} = \sup_{x \in E, \| x \|_E \leq 1} \| Ax \|_F$. It follows from [5] that $P_N \in \mathcal{L}(C[t_{j-1}, t_j], C[t_{j-1}, t_j])$ ($j = 1, \ldots, N$) and $P_N \in \mathcal{L}(C[0, b], L^\infty(0, b))$. Moreover, the norms of these operators are uniformly bounded in $N$:

$$\max_{j=1, \ldots, N} \| P_N \|_{\mathcal{L}(C[t_{j-1}, t_j], C[t_{j-1}, t_j])} \leq c, \quad N \in \mathbb{N}, \quad (11)$$

Here $c$ is a positive constant which is independent of $j$ and $N$. On the basis of (11) we obtain that

$$\| u - P_N u \|_{L^\infty(0, b)} \rightarrow 0 \quad \text{as} \quad N \rightarrow \infty \quad (12)$$

for every $u \in C[0, b]$. A consequence of this is

**Lemma 2.** Let $S : L^\infty(0, b) \rightarrow C[0, b]$ be a linear compact operator. Then

$$\| S - P_N S \|_{\mathcal{L}(L^\infty(0, b), L^\infty(0, b))} \rightarrow 0 \quad \text{as} \quad N \rightarrow \infty. \quad (13)$$

In the following we present a result about the rate of the error $\| u - P_N u \|_{L^\infty(0, b)}$.

**Lemma 3.** Let $u \in C_{d,\nu}^m[0, b], m \in \mathbb{N}, -\nu < \nu < 1$, $p \in \{0, 1, \ldots, m\}$. Let the node points (8) with grid points (5) and parameters (9) be used. Let $P_N : C[0, b] \rightarrow S_{m-1}^{(-1)}(\Delta_N)$ be determined by the conditions (10).

Then

$$\| u - P_N u \|_{L^\infty(0, b)} \leq c \varepsilon_N, \quad (13)$$

where $c$ is a positive constant not depending on $N$ and $\varepsilon_N = \varepsilon_N(m, \nu, p, r, r_d)$ is defined as follows:

$$\varepsilon_N = N^{-m}, \quad (14)$$
\[ m < 1 - \nu, \ p \geq 0, \ r \geq 1, \ r_d \geq 1; \]
\[ m = 1 - \nu, \ p = 0, \ r > 1, \ r_d > 1; \]
\[ m = 1 - \nu, \ p > 0, \ r > 1, \ r_d \geq 1; \]
\[ 1 - \nu < m < 1 - \nu + p, \ p > 0, \]
\[ r \geq \frac{m}{1 - \nu}, \ r_d \geq 1; \]
\[ m = 1 - \nu + p, \ p > 0, \ r \geq \frac{m}{1 - \nu}, \ r_d > 1; \]
\[ m > 1 - \nu + p, \ p \geq 0, \]
\[ 1 \leq r < \frac{m}{1 - \nu}, \ r_d \geq \frac{m}{1 - \nu + p}; \]
\[ \varepsilon_N = N^{-r(1 - \nu)} \] (15)
\[ 1 - \nu < m < 1 - \nu + p, \ p > 0, \]
\[ 1 \leq r < \frac{m}{1 - \nu}, \ r_d \geq \frac{m}{1 - \nu + p}; \]
\[ \varepsilon_N = N^{-\min\{r(1 - \nu), r_d(1 - \nu + p)\}} \] (17)
\[ m > 1 - \nu + p, \ p \geq 0, \]
\[ 1 \leq r < \frac{m}{1 - \nu}, \ 1 \leq r_d < \frac{m}{1 - \nu + p}; \]
\[ \varepsilon_N = N^{-r_d(1 - \nu + p)} \] (18)

for
\[ m > 1 - \nu + p, \ p \geq 0, \]
\[ r \geq \frac{m}{1 - \nu}, \ 1 \leq r_d < \frac{m}{1 - \nu + p}. \]

Proof. We follow the approach and techniques of [5]. It follows from (11) that
\[ \|u - P_N u\|_{L^\infty(0, b)} \leq c \max_{j=1, \ldots, N} \max_{t_{j-1} \leq t \leq t_j} |u(t) - v(t)|, \]

where \(c\) is a positive constant not depending on \(N\) and \(v\) is an arbitrary element of the space \(S_m^{-1}(\Delta_N)\). Thus, in order to study the rate of the error \(\|u - P_N u\|_{L^\infty(0, b)}\), we have to estimate \(|u(t) - v(t)|\) for a suitable \(v(t)\) on every subinterval \([t_{j-1}, t_j], j = 1, \ldots, N = 4n\). In particular, taking
\[ v(t) = u(t_j) + u'(t_j)(t-t_j) + \frac{1}{2!}u''(t_j)(t-t_j)^2 + \ldots + \frac{1}{(m-1)!}u^{(m-1)}(t_j)(t-t_j)^{m-1}, \]

where \(t \in [t_{j-1}, t_j], j = 1, \ldots, n\), and using (4) for the derivatives of \(u \in C_{m, \nu}^{d, p}[0, b]\), we can estimate \(u(t) - v(t)\) on the subinterval \([t_{j-1}, t_j] \subset [0, \frac{b}{4}], j = 1, \ldots, n\); in a similar way we can derive the estimates for \(u - v\) (with a suitable \(v \in S_m^{-1}(\Delta_N)\)) on other subintervals \([t_{j-1}, t_j] \subset [\frac{b}{4}, b], j = n + 1, \ldots, 4n\); see [2] for a detailed proof.

4 Collocation method
We look for an approximation \(u_N\) to the solution \(u\) of equation (1) determining \(u_N\) from the following conditions:
\[ u_N(t) - \int_0^b K(t,s)u_N(s)ds - f(t) = 0, \quad t = \xi_i, p \]
\[ u_N \in S_m^{-1}(\Delta_N), \quad m \geq 1, \]
\[ p = 1, \ldots, m; \ i = 1, \ldots, N, \] (19)

with \(\{\xi_{i, p}\}\), given by (8).

Theorem 1. Let the following conditions be fulfilled:
1) \(K \in C_{d, 0}^{1, \nu}[0, b]\), \(\nu < 1, 0 < d < b\);
2) \(f \in C[0, b]\);
3) the homogeneous integral equation
\[ u(t) = \int_0^b K(t,s)u(s)ds, \quad 0 \leq t \leq b, \] (20)

has only the trivial solution \(u = 0\);
4) the collocation points (8) with grid points (5) and parameters (9) are used.
Then equation (1) has a unique solution \( u^* \in C[0,b] \). For all sufficiently large \( N \), say \( N \geq N_0 \), the collocation conditions (19) determine for every choice of parameters \(-1 \leq \eta_1 < \ldots < \eta_m \leq 1\) a unique approximation \( u^*_N \in S_{m-1}^{(-1)}(\Delta_N) \) to \( u^* \) and

\[
\sup_{t \in [0,b]} |u^*_N(t) - u^*(t)| \to 0 \quad \text{as} \quad N \to \infty. \quad (21)
\]

**Proof.** We consider equation (1) as the equation

\[
u = Tu + f
\]

in the Banach space \( L^\infty(0,b) \), with the operator \( T \), defined by \( (Tu)(t) = \int_0^b K(t,s)u(s)\,ds \). It follows from the assumption 1) that \( T \) is compact as an operator from \( L^\infty(0,b) \) to \( C[0,b] \) and from \( L^\infty(0,b) \) to \( L^\infty(0,b) \), also. Since equation \( u = Tu \) has only the trivial solution \( u = 0 \), then there exists the inverse operator \( (I - T)^{-1} \in \mathcal{L}(L^\infty(0,b), L^\infty(0,b)) \) and equation (22) has a unique solution \( u^* = (I - T)^{-1}f \in L^\infty(0,b) \). Since \( f \in C[0,b] \) and \( T \in \mathcal{L}(L^\infty(0,b), C[0,b]) \), then \( u^* \in C[0,b] \).

The collocation conditions (19) can be written in the form

\[
u_N = P_N Tu_N + P_N f,
\]

with \( P_N : C[0,b] \to S_{m-1}^{(-1)}(\Delta_N) \), defined in Sec. 3. By Lemma 2,

\[
\|T - P_N T\|_{\mathcal{L}(L^\infty(0,b), L^\infty(0,b))} \to 0 \quad \text{for} \quad N \to \infty.
\]

Using (24) we obtain that \( (I - P_N T) \) is invertible for all sufficiently large \( N \), say \( N \geq N_0 \), and

\[
\|(I - P_N T)^{-1}\|_{\mathcal{L}(L^\infty(0,b), L^\infty(0,b))} \leq c, \quad N \geq N_0,
\]

where \( c \) is a positive constant which is independent of \( N \). This shows that for \( N \geq N_0 \) equation (23) has a unique solution \( u^*_N = (I - P_N T)^{-1}P_N f \). We have for it and \( u^* \), the solution of equation (22),

\[
(I - P_N T)(u^* - u^*_N) = (I - P_N T)u^* - (I - P_N T)u_N = u^* - P_N Tu^* - P_N f = u^* - P_N f - (P_N u^* - P_N f) = u^* - P_N u^*.
\]

Therefore,

\[
u^* - u^*_N = (I - P_N T)^{-1}(u^* - P_N u^*).
\]

Taking the norms and using (25), we have

\[
\|u^* - u^*_N\|_{L^\infty(0,b)} \leq c\|u^* - P_N u^*\|_{L^\infty(0,b)}, \quad N \geq N_0,
\]

where \( c \) is a constant which is independent of \( N \). Since \( u^* \in C[0,b] \), the convergence (21) follows from (12) and (26).

**Theorem 2.** Let the following conditions be fulfilled:

1) \( K(t,s) = g(t,s)\kappa(t,s) \) is subject to the conditions, stated in the assumptions (A1) and (A2) (see Sec. 2);

2) \( f \in C_{d,p}^{m,\nu}[0,b] \), with \( m, \nu, d, p \), fixed in (A1) and (A2);

3) equation (20) has only the trivial solution \( u = 0 \);

4) the collocation points (8) with grid points (5) and parameters (9) are used.

Then for all sufficiently large \( N \), say \( N \geq N_0 \), the collocation conditions (19) determine for every choice of parameters \(-1 \leq \eta_1 < \ldots < \eta_m \leq 1\) a unique approximation \( u^*_N \in S_{m-1}^{(-1)}(\Delta_N) \) to \( u^* \), the exact solution of equation (1). The following error estimate holds:

\[
\sup_{0 \leq t \leq b} \left| u^*(t) - u^*_N(t) \right| \leq c\varepsilon_N, \quad N \geq N_0,
\]

where \( c \) is a positive constant not depending on \( N \) and \( \varepsilon_N \) is defined by the formulas (14)-(18).

**Proof.** Due to Theorem 1 we have to prove only the estimate (27). By Lemma 1, \( u^* \in C_{d,p}^{m,\nu}[0,b] \). Now the estimate (27) follows from Lemma 3 and the inequality (26).

### 5 Superconvergence phenomenon

Theorem 2 suggests that by using a collocation method based on piecewise polynomials of degree \( m - 1 \) \((m \geq 1)\) and graded grids of type (5), one can reach a convergence order

\[
\sup_{0 \leq t \leq b} \left| u^*(t) - u^*_N(t) \right| \leq cN^{-m}, \quad N \geq N_0
\]

for sufficiently large values of grid parameters \( r \) and \( r_d \), see (14)-(18) and (27).
In (28) the order \( m \) cannot be improved, whereas piecewise polynomials of the order \( m - 1 \) are used for the approximation. Nevertheless, as it will be seen from Theorem 3 below, the convergence order at the collocation points will be higher than \( O(N^{-m}) \) for a special choice of collocation parameters (9). Actually, we shall assume that the points (9) are the nodes of a quadrature formula

\[
\int_{-1}^{1} g(s) \, ds = \sum_{k=1}^{m} w_k g(\eta_k) + R_m(g),
\]

\( -1 \leq \eta_1 < \ldots < \eta_m \leq 1, \)

which is exact for all polynomials of degree \( m \).

Note that the weights \( w_k \) \( (k = 1, \ldots, m) \) will not be used in our algorithms. The existence of a quadrature formula (29) which is exact for polynomials of degree \( m \) is used in the proof of the following

**Theorem 3.** Let \( \nu \in \mathbb{R}, \nu < 1, m \in \mathbb{N}, \)

\( 0 < \rho < 1, \rho \in \{0, 1, \ldots, m+1\}. \) Assume that the following conditions are fulfilled.

(i) The kernel \( K(t,s) = g(t,s)\kappa(t,s) \) in equation (1) satisfies the conditions (A1) and (A2) with \( m+1 \) instead of \( m \).

(ii) \( f \in C_{d,p}^{m+1,\nu}[0,b]. \)

(iii) The integral equation (20) has only the trivial solution \( u = 0. \)

(iv) The collocation points (8) with grid points (5) and parameters (9) are used, where \( r \) and \( r_d \) are chosen so that

- if \( m < 1 - \nu, \rho \geq 0, \) then \( r \geq 1, r_d \geq 1; \)
- if \( m = 1 - \nu, \rho = 0, \) then \( r > 1, r_d > 1; \)
- if \( m = 1 - \nu, \rho \geq 1, \) then \( r > 1, r_d \geq 1; \)
- if \( 1 - \nu + p > m > 1 - \nu, p \geq 1, \) then \( r \geq \frac{m}{1 - \nu}, r_d \geq 1; \)
- if \( m = 1 - \nu + p, p \geq 1, \) then \( r \geq \frac{m}{1 - \nu}, r_d > 1; \)
- if \( m > 1 - \nu + p, p \geq 0, \) then \( r \geq \frac{m}{1 - \nu}, r_d \geq \frac{m}{1 - \nu + p}. \)

(v) The quadrature formula (29) is exact for all polynomials of degree \( m. \)

Then for all sufficiently large \( N, \) say \( N \geq N_0, \)

the collocation conditions (19) determine a unique approximation \( u_N^* \in S_{m-1}^{(m-1)}(\Delta_N) \) to \( u^* \in C[0,b], \)

the exact solution of equation (1). For \( N \geq N_0, \)

the following error estimate holds:

\[
\max_{q=1,\ldots,m;\xi=1,\ldots,N} \left| u_N^*(\xi,q) - u^*(\xi,q) \right| \leq \begin{cases} 
N^{-1}, & \text{if } \nu < 0, \\
N^{-1} \ln N, & \text{if } \nu = 0, \\
N^{-(1-\nu)}, & \text{if } \nu > 0.
\end{cases}
\]

Here \( c \) is a positive constant which is independent of \( N. \)

**Proof.** See [2,3].

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**References**


