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Nonlinear Wavelet Packets Algorithms

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Abstract: A nonlinear multiresolution-packets is presented. The stability and the improvements of the new algorithm are studied. The goal is to describe tools for adapting methods of analysis to various tasks occurring in harmonic analysis and signal processing.

Key Words: Multi-scale decomposition, nonlinear reconstruction, stability, wavelets -packets.

1 Introduction

We would like to describe a method permitting efficient compression of a variety of signals such as sound and images. The method can use any linear or nonlinear multiresolution. In the first case, we can recover the biorthogonal wavelet-packets and the interpolating wavelet-packets, but in the second case a new algorithm is obtained. The concept of wavelet-packets has been introduced by R.R.Coifman et al. [6],[7] as a generalization of wavelet bases. There are several applications of these representations as: image analysis [14], data compression [6], adaptive methods for approximation of nonstationary partial differential equations [13],...

Even in the linear case, decomposition and reconstruction algorithms are obviously nonlinear transformations to represent a signal in its own best multiresolution. We modify the direct algorithms in order to ensure stability. We introduce some error-control algorithms for different multiresolutions.

The paper is organized as follows: We recall Harten's multiresolution framework. We introduce the general multiresolution-packets in section three. Special attention is paid to error-control in next section.

2 Harten's framework

The discrete multiresolution framework introduced by Harten is based on two op-

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erators: decimation and prediction.

$$D_k^{k-1} : V^k \to V^{k-1}, \tag{1}$$

$$P_{k-1}^k : V^{k-1} \to V^k. \tag{2}$$

From a set of discrete data $f^k = (f_i^k)_{i=1}^{N_k}$, where k represents the discretization level, the decimation operator D_k^{k-1} computes $f^{k-1} = (f_i^{k-1})_{i=1}^{N_{k-1}}$, at the next coarser discretization level ($N_{k-1} < N_k$). The prediction operator made an approximation $\tilde{f}^k = (\tilde{f}_i^k)_{i=1}^{N_k}$ to $f^k = (f_i^k)_{i=1}^{N_k}$ from $f^{k-1} = (f_i^{k-1})_{i=1}^{N_{k-1}}$.

The decimation operator is always assumed to be linear. In contrast, the prediction operator need not be linear, but should at least satisfy the consistency requirement $D_k^{k-1} \cdot P_{k-1}^k = I_{N_{k-1}}$, where $I_{N_{k-1}}$ denotes the identity operator in $IR^{N_{k-1}}$. If a nonlinear operator is considered as prediction we will obtain a nonlinear multiresolution. From the consistency property, it follows that the null space of D_k^{k-1} has dimension $N_k - N_{k-1}$, since the image of D_k^{k-1} is the full $IR^{N_{k-1}}$. Then we can decompose the prediction error according to

$$f^k - \tilde{f}^k = \sum_{i=1}^{N_k - N_{k-1}} d_i^{k-1} b_i^{k-1} \tag{3}$$

where $(b_i^{k-1})_{i=1}^{N_k - N_{k-1}}$ is a basis of W_k (space of the details defined as the null space of the prediction operator).

By iteration of this process from $k = L$ to $k = 1$, we obtain a multiscale decomposition of f^L into $(f^0, d^0, d^1, \ldots, d^{L-1})$.

Let G_k be the operator which computes the coordinates of the prediction error in a basis of $\mathcal{N}(D_k^{k-1})$, E_k such that $e^k = E_k G_k e^k$. Then the direct and inverse transforms of the multiresolution process take the form

$$v^L \quad \to \quad Mv^L \quad \text{(Encoding)}$$

$$\begin{cases} \text{Do} \quad k = L, \ldots, 1 \\ v^{k-1} = D_k^{k-1} v^k \\ d^k = G_k(v^k - P_{k-1}^k v^{k-1}) \end{cases} \tag{4}$$

$$Mv^L = \{v^0, d^1, \ldots, d^L\}$$

$$Mv^L \quad \to \quad M^{-1}Mv^L \quad \text{(Decoding)}$$

$$\begin{cases} \text{Do} \quad k = 1, \ldots, L \\ v^k = P_{k-1}^k v^{k-1} + E_k d^k \end{cases} \tag{5}$$

On the other hand, in practice the prediction operator (the decimation operator also) is constructed by using two fundamental tools: discretization and reconstruction. The discretization $\mathcal{D}_k$ is a linear operator that connects a functional space $\mathcal{F}$ with the space V^k and yields discrete information at the resolution level k specified by a grid X^k. The reconstruction operator $\mathcal{R}_k$ goes from V^k to $\mathcal{F}$. A basic consistency requirement is that

$$\mathcal{D}_k \mathcal{R}_k f^k = f^k \tag{6}$$

Given sequences of discretization and reconstruction operators satisfying (6), it is then possible to define the decimation and prediction operators according to

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$$D^k_{k-1} = \mathcal{D}_{k-1}\mathcal{R}_k. \quad (7)$$

$$P^{k-1}_k = \mathcal{D}_k\mathcal{R}_{k-1}. \quad (8)$$

3 Multiresolution packets

In this section we shall introduce the general "Multiresolution packet". The same as the library of wavelet packet bases it is naturally organized as subsets of binary tree. This segmentation of signals into those dyadic intervals is better adapted to the frequency content. The idea is to obtain the best decomposition of all the possible ones. We now define a cost function on sequence and search for its minimum over all representation in a library. For a given vector, their minima are the most efficient representation.

Definition 1 *A map $\mathcal{L}$ from sequences $\{x_j\}$ to R is called an additive information cost function if $\mathcal{L}(0) = 0$ and $\mathcal{L}(\{x_j\}) = \sum_j \mathcal{L}(x_j)$.*

Some useful examples of information cost include: a) Number above a threshold, set an arbitrary threshold ϵ and count the elements in the sequence x whose absolute value exceeds ϵ. b) Concentration in l^p norm ($p < 2$), $\mathcal{L}(x) = ||x||_p$. c) Entropy, $\mathcal{L}(x) = -\sum_j p_j log p_j$ where $p_j = \frac{|x_j|^2}{||x||^2}$ and we set $p\ log p = 0$ if $p = 0$. d) Logarithm of energy, $\mathcal{L}(x) = \sum_j log|x_j|^2$. For more details see [8]. Here we use the first possibility.

As the library is a tree, then we can find the best representation by induction on the number of scales. Denote by s^k_j the representation of vectors corresponding to the scale k, $j = 1, 2, \ldots, 2^{L-k}$, and by $\mathcal{B}^k_j$ the best representation for x. For $k = L$, $\mathcal{B}^L = s^L$. We construct $\mathcal{B}^{k-1}_j = s^{k-1}_j$ if $\mathcal{L}(\mathcal{B}^k_j) > \mathcal{L}(s^{k-1}_{2j}) + \mathcal{L}(s^{k-1}_{2j+1})$ and $\mathcal{B}^{k-1}_j = \mathcal{B}^k_{2j} + \mathcal{B}^k_{2j+1}$ otherwise.

Whenever a parent node is of lower information cost than the children, we mark the parent. In the final representation we have all the information, that is, the value of the details and the marks.

In practice, we start with a vector of data $s^L_1 = f^L$, corresponding to any discretization of a certain function. We compute a step of the multiresolution algorithm, that is, $s^{L-1}_1 = f^{L-1}$ and the details $s^{L-1}_2 = d^L$. If the addition of the cost of these two new vectors are higher than it comes from s^L_1 we do not consider the decomposition. On the other hand, if the cost is minor then we carry out the decomposition. If the last case has been produced then we would repeat the process for these two new vectors (s^{L-1}_1 and s^{L-1}_2) independently. Anyhow, the decomposition is finished when one has arrived to the worst resolution level prescribed by the user.

In the framework of Harten, the one to one correspondence between two discretization levels, when $\mathcal{L}(\mathcal{B}^k_j) > \mathcal{L}(s^{k-1}_{2j}) + \mathcal{L}(s^{k-1}_{2j+1})$, is given by

$$s^{k-1}_j = \begin{cases} D^{k-1}_k(s^k_{\frac{j+1}{2}}) & j \text{ ood} \\ G_k Q_k(s^k_{\frac{j}{2}}) & \text{otherwise} \end{cases} \quad (9)$$

$$s^k_j = P^k_{k-1}(s^{k-1}_{2j-1}) + E_k(s^{k-1}_{2j}) \quad (10)$$

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4 Error-control algorithms

In this section we are going to study the stability concept in the multiresolution-packets framework. We consider modified algorithms. Next, we shall introduce some results of stability associated to the modified algorithms. We can consider different multiresolutions (linear and non linear) for the scales and the details. Our modified algorithms have a similar structure than the original algorithms introduced by Harten for the multiresolution framework [10] (see [1] for the 2-D case).

Modified encoding procedure for the point values multiresolution:

Algorithm 1

for $k = L, \ldots, 1$
  for $j = 0, \ldots, J_{k-1}$
    $\bar{f}^{k-1}_j = \bar{f}^k_{2j}$
  end
end
Set $\hat{f}^0 = \bar{f}^0$
for $k = 1, \ldots, L$
  $\hat{f}^k_0 = \bar{f}^k_0$
  for $j = 1, \ldots, J_{k-1}$
  $f^P_{2j-1} = (P^k_{k-1}\hat{f}^{k-1})_{2j-1}$
  $\bar{d}^k_j = \bar{f}^k_{2j-1} - f^P_{2j-1}$
  end
  $\hat{e}^k = tr(MP(\bar{d}^k, \epsilon_k))$
  $\hat{d}^k = (\hat{M}\hat{P})^{-1}\hat{e}^k$
  for $j = 1, \ldots, J_{k-1}$
  $\hat{f}^k_{2j-1} = f^P_{2j-1} + \hat{d}^k_j$
  $\hat{f}^k_{2j} = \hat{f}^{k-1}_j$
  end
end
  $M^M\bar{f}^L = \{\bar{f}^0, \hat{e}^1, \ldots, \hat{e}^L\}$

Remark 1 *Whenever a parent node is of lower information cost than the children, we mark the parent. In the representation* $\{\hat{e}^1, \ldots, \hat{e}^L\}$ *we have all the information, that is, the value of the details and the marks. With* $(\hat{M}\hat{P})^{-1}$ *we recover some approximation of the value* $\bar{d}^k$.

Using this algorithm we can control the final error, in particular we obtain stability properties. Notice you that the computational cost of standard encoding and E-C encoding are the same.

Proposition 1 *Given a discrete sequence* f^L *and a tolerance level* ϵ*, if the truncation parameters* ϵ_k*, in the modified encoding algorithm are chosen so that*

$$\epsilon_k := \epsilon$$

then the sequence $\hat{f}^L$ *satisfies*

$$||f^L - \hat{f}^L||_p \leq \epsilon \qquad (11)$$

for $p = \infty, 1$ *and* 2*. Thus, the modified algorithm for the interpolatory case is stable.*

Using a similar error-control algorithm, we can obtain the following bounds in the cell-average framework.

Proposition 2 *Given a discrete sequence* f^L *and a tolerance level* ϵ*, if the truncation parameters* ϵ_k *in the modified encoding algorithm are chosen so that*

$$\epsilon_k := \epsilon \cdot (\frac{1}{2})^{L-k}$$

then the sequence $\hat{f}^L$ *satisfies*

$$||f^L - \hat{f}^L||_p \leq C\epsilon \qquad (12)$$

for $p = \infty, 1$ *and* 2. $C = 2$ *for* $p = \infty, 1$ *and* $C = \frac{2}{\sqrt{3}}$ *for* $p = 2$. *Thus, the modified algorithm for the cell average case is stable.*

Remark 2 *Those propositions give us explicit bounds of the error.*

Remark 3 *As we said before, other measure of a sequence is the* l^2logl^2 *norm:*

$$\mathcal{L}(x) = -\sum_j |x_j|^2 ln|x_j|^2 \qquad (13)$$

and the threshold in the details is $\sqrt{\epsilon \cdot exp(\frac{-\mathcal{L}(x)}{||x||^2})}$. *The term* $exp(\frac{-\mathcal{L}(x)}{||x||^2})$ *is directly related to average energy of significant coefficients. With the same ideas we can obtain propositions in this context.*

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