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Cell-average Nonlinear Multiresolution on the Quincunx Pyramid

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Abstract: This paper links the non separable quincunx pyramid and the nonlinear discrete cell-average Harten's multiresolution framework. In order to obtain the stability of these representations, some modified multiresolution processing algorithms are introduced. A prescribed accuracy in various discrete norms is ensured by these modified strategies. Explicit error bounds are presented.

Key Words: Nonlinear multiresolution, quincunx pyramid, error-control, image processing.

1 Introduction

Multi-scale representations, as Harten's multiresolutions [12], are one of the most efficient tools for image compression. In the multiresolution transforms a discrete sequence $\bar{f}^L$ which represent sampling of weighted-averages of a function $f(x)$ at the finest resolution level L is encoded to produce a multi-scale representation of its information contents $(\bar{f}^0, e^1, e^2, \ldots, e^L)$, where the $\bar{f}^0$ corresponds to the sampling at the coarsest resolution level and each sequence e^k represent the intermediate details which are necessary to recover $\bar{f}^k$ from $\bar{f}^{k-1}$. This representation of the signal is well adapted to data compression procedures.

Thus, the multi-scale representation is processed (truncation and/or quantization) and the result of this step gives a modified multi-scale representation $(\hat{f}^0, \hat{e}^1, \hat{e}^2, \ldots, \hat{e}^L)$ which is *close* to the original one. After decoding the processed representation, we obtain a discrete set $\hat{f}^L$ which is expected to be *close* to the original discrete set $\bar{f}^L$. In order for this to be true, some form of stability is needed.

Harten's framework was developed to use nonlinear reconstruction processes. In image examples [3],[4],[5],[7] we can see the nonlinear process allows a better adapted treatment of edges, in the sense that they do not generate so many large detail coefficients as in the standard linear wavelet transforms. Several nonlinear wavelets transforms can be found in [8]-[10]-[14]-[16].

In the nonlinear case, stability can be ensured by modifying the encoding algorithm. The idea of a modified-encoding to deal with nonlinear multiresolution schemes is due to Harten; one dimensional algorithms in several settings can be found in [6], [11]. We modify the direct transform in such a way that the error accumulated in processing the values of the originally multi-scale representation remains under a prescribed value. The goal of this procedure is to keep track of the accumulation error in processing the values in the multi-scale representation.

The natural extension to the multiresolution analysis of images is based on the tensor product of two 1-D multiresolutions. The associated algorithm uses the separable pyramid. However in many practical applications much better results can be achieved by the use of nonseparable extensions. One nonseparable decomposition having interesting properties is the quincunx pyramid. We refer [9]-[13]-[15] for advances including nonlinear techniques in the wavelet framework.

The aim of this paper is to present stable nonlinear two-dimensional cell-average multiresolution algorithms based on the quincunx pyramid. We generalize our previous work in the point-values framework that is not good adapted for image processing [4].

2 Harten's framework

Harten's framework is based on two fundamental tools: discretization $\mathcal{D}_k$ and reconstruction $\mathcal{R}_k$. The discretization operator obtains discrete information from a (non-discrete) signal ($f \in \mathcal{F}$) at a particular resolution level k. The reconstruction operator, on the other hand, produces an approximation to a signal from its discrete values. This reconstruction can be *nonlinear*, and then better adapted to the considered problem.

Using these two operators we can connect linear vector spaces, V^k, that represent in some way the different resolution levels (k increasing implies more resolution), i.e.,

$$\begin{aligned} D_k^{k-1} \;&:\; V^k \to V^{k-1}, \quad \textit{decimation}, \\ P_{k-1}^k \;&:\; V^{k-1} \to V^k, \quad \textit{prediction}. \end{aligned}$$

2.1 Cell-average MR analysis on the Quincunx pyramid

In this subsection, we present how to define a Harten's multiresolution analysis in $[0,1] \times [0,1]$ which admits a quincunx pyramid as a decomposition algorithm.

The transform $T(x,y) = (x+y, x-y)$ defines the sub-sampling grid of the quincunx pyramid. Note that $T^2 = 2Id$, which is the basic sub-sampling of the dyadic algorithm. Thus, in practice, the finest resolution level L is considered even.

Let $X^L = \{x_i^L, y_j^L\}_{i,j=0}^{J_L}$, $x_i^L = ih_L$, $y_j^L = jh_L$, $h_L = 2^{-L}h_0$, $J_L = 2^L J_0$, J_0 some integer, $h_0 = \frac{1}{J_0}$, L even.

Since $T^2 = 2Id$, we obtain, for $i,j = 0,\ldots,\frac{J_L}{2}$, $x_{2i}^L = x_i^{L-2}$ and $y_{2j}^L = y_j^{L-2}$.

In this case we have

$$\mathcal{D}_k : L^1([0,1]\times[0,1]) \longrightarrow V^k, \tag{1}$$

$$\bar{f}_{i,j}^k = (\mathcal{D}_k f)_{i,j} = \frac{1}{|\Omega_{i,j}^k|}\int_{\Omega_{i,j}^k} f(x,y)dydx, \tag{2}$$

where $L^1([0,1]\times[0,1])$ is the space of absolutely integrable functions in $[0,1]\times[0,1]$ and

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for k even

$$\Omega_{i,j}^k = [x_{i-1}^k, x_i^k] \times [y_{j-1}^k, y_j^k],$$

for k odd and j odd

$$\Omega_{i,j}^k = \{(x,y) : y_{j-1}^k \le y \le y_{j+1}^k$$

$$x_{i+1}^k + \frac{x_{i+1}^k - x_{i-1}^k}{y_{j+1}^k - y_{j-1}^k}(y - y_{j+1}^k) \le x \le x_{i+1}^k\},$$

and finally for k odd and $j+1$ even

$$\Omega_{i,j+1}^k = \{(x,y) : y_{j-2}^k \le y \le y_j^k$$

$$x_{i-1}^k \le x \le x_{i-1}^k + \frac{x_{i+1}^k - x_{i-1}^k}{y_j^k - y_{j-2}^k}(y - y_{j-2}^k)\}.$$

This analysis turns out to be appropriate for data compression of discontinuous, piecewise smooth signals.

It is sufficient to consider weighted averages $\bar{f}_{i,j}^k$ for $1 \le i,j \le J_k$ for k even and $1 \le i \le J_k$ and $1 \le j \le 2J_k$ for k odd since these contain information on f over $[0,1] \times [0,1]$. Thus, V^k is the space of sequences with $J_k \times J_k$ components for k even and $J_k \times 2J_k$ for k odd.

Moreover, for k even

$$\bar{f}_{i,j}^{k-2} = \frac{1}{4}(\bar{f}_{2i-1,2j-1}^k + \bar{f}_{2i-1,2j}^k + \bar{f}_{2i,2j-1}^k + \bar{f}_{2i,2j}^k)$$

$i,j = 1,2,\ldots,J_{k-2}$.

$$\bar{f}_{i,j}^{k-2} = \frac{1}{2}(\bar{f}_{i,2j-1}^{k-1} + \bar{f}_{i,2j}^{k-1})$$

$i,j = 1,2,\ldots,J_{k-2}$.

Finally, for $i,j = 1,2,\ldots,J_{k-1}$,

$$\bar{f}_{i,2j-1}^{k-1} = \frac{1}{2}(\bar{f}_{2i,2j-1}^k + \frac{\bar{f}_{2i-1,2j-1}^k + \bar{f}_{2i,2j}^k}{2}),$$

and

$$\bar{f}_{i,2j}^{k-1} = \frac{1}{2}(\bar{f}_{2i-1,2j}^k + \frac{\bar{f}_{2i-1,2j-1}^k + \bar{f}_{2i,2j}^k}{2}).$$

On the other hand, taking k even, since

$$0 = \frac{1}{2}(e_{2i,2j-1}^k + \frac{e_{2i-1,2j-1}^k + e_{2i,2j}^k}{2}),$$

$$0 = \frac{1}{2}(e_{2i-1,2j}^k + \frac{e_{2i-1,2j-1}^k + e_{2i,2j}^k}{2}),$$

and

$$0 = (e_{i,2j-1}^{k-1} + e_{i,2j}^{k-1})/2,$$

we will keep $e_{2i,2j}^k, e_{2i-1,2j-1}^k, e_{i,2j-1}^{k-1}$ only.

A reconstruction operator for this discretization is any operator $\mathcal{R}_k$ satisfying

$$\mathcal{R}_k : V_k \longrightarrow L^1([0,1] \times [0,1]),$$

$$\begin{aligned}(\mathcal{D}_k \mathcal{R}_k \bar{f}^k)_{i,j} &= \frac{1}{|\Omega_{i,j}^k|} \int_{\Omega_{i,j}^k} (\mathcal{R}_k \bar{f}^k)(x,y)dxdy \\ &= \bar{f}_{i,j}^k.\end{aligned}$$

That is, $\mathcal{R}_k \bar{f}^k(x,y)$ has to be a function in $L^1([0,1] \times [0,1])$ whose mean value on the (i,j)-th cell coincides with $\bar{f}_{i,j}^k$, $\forall i,j$. Finally, $P_{k-1}^k := \mathcal{D}_k \mathcal{R}_{k-1}$.

A possibility to find the desired reconstruction is using the primitive function. In our case, $|\Omega^k| = |\Omega_{i,j}^k|$, then we consider

$$F_{i,j}^k = |\Omega^k| \sum_{l=1}^{i} \sum_{m=1}^{j} \bar{f}_{l,m}^k$$

If we consider $I_k((x,y);F^k)$ an interpolatory reconstruction of the primitive function then $(\mathcal{R}_k \bar{f}^k)(x,y) := \frac{d}{dx}\frac{d}{dy} I_k((x,y);F^k)$ will be a reconstruction for the original function.

3 Compression transformations using error control

Multiresolution representations lead naturally to data-compression algorithms.

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By applying the inverse multiresolution transform to the compressed representation, we obtain $\hat{f}^L = M^{-1}\{\hat{f}^0, \hat{e}^1, \ldots, \hat{e}^L\}$, an approximation to the original signal $\bar{f}^L$. We expect the information contents of $\hat{f}^L$ to be close to those of the original signal $\bar{f}^L$, and in order for this to be true, the stability of the multiresolution scheme with respect to perturbations is essential.

Given a discrete sequence $\bar{f}^L$ and a tolerance level ϵ for accuracy, our task is to come up with a compressed representation

$$\{\hat{f}^0, \hat{e}^1, \ldots, \hat{e}^L\} \tag{3}$$

such that if $\hat{f}^L = M^{-1}\{\hat{f}^0, \hat{e}^1, \ldots, \hat{e}^L\}$, we have

$$\| \bar{f}^L - \hat{f}^L \| \leq C\epsilon \tag{4}$$

for an appropriate norm.

As observed by Harten [11], one possible way to accomplish this goal is to modify the encoding procedure in such a way that the modification allows us to keep track of the cumulative error and truncate accordingly.

In what follows we present a quincunx cell-average extension of the one dimensional algorithms presented in [6], [11], the two dimensional tensor product in [2] and the quincunx point-values in [4].

The modified encoding procedure enables us to specify the desired level of accuracy in the decompressed signal. A modified encoding procedure is designed keeping in mind the particular decoding procedure to be used.

The modified algorithm for the cell-average is similar to the point-values, changing the details we process. In this case, for k even

$$\begin{aligned} \hat{e}^{k-1}_{i,\cdot} &= \mathbf{pr}(\bar{f}^{k-1}_{i,\cdot} - (P^k_{k-1}\hat{f}^{k-2})_{i,\cdot} \\ &\quad -(f^{k-2}_{i,j} - \hat{f}^{k-2}_{i,j}), \epsilon_{k-1}) \end{aligned}$$

and

$$\begin{aligned} \hat{e}^{k}_{\cdot,\cdot} &= \mathbf{pr}(\bar{f}^{k}_{\cdot,\cdot} - (P^k_{k-1}\hat{f}^{k-1})_{\cdot,\cdot} \\ &\quad -(f^{k-1}_{i,\cdot} - \hat{f}^{k-1}_{i,\cdot}), \epsilon_{k}) \end{aligned}$$

where **pr** denotes the process step (truncation and/or quantization).

4 Stability and explicit error bounds

We use the following norms:

$$\begin{aligned} \left\|\bar{f}^k\right\|_\infty &= \sup_{i,j}|\bar{f}^k_{i,j}|, \\ \left\|\bar{f}^k\right\|_1 &= \frac{1}{dim(V_k)}(\sum_{i,j}|\bar{f}^k_{i,j}|), \\ \left\|\bar{f}^k\right\|_2^2 &= \frac{1}{dim(V_k)}(\sum_{i,j}|\bar{f}^k_{i,j}|^2), \end{aligned}$$

where $\bar{f}^k = \{\bar{f}^k_{ij}\}$.

We denote for k even

$$\begin{aligned} \hat{e}^{k-1}_{i,\cdot} &= \mathbf{pr}(\bar{f}^{k-1}_{i,\cdot} - (P^k_{k-1}\hat{f}^{k-2})_{i,\cdot} \\ &\quad -(f^{k-2}_{i,j} - \hat{f}^{k-2}_{i,j}), \epsilon_{k-1}) \end{aligned}$$

$$\begin{aligned} \hat{e}^{k}_{\cdot,\cdot} &= \mathbf{pr}(\bar{f}^{k}_{\cdot,\cdot} - (P^k_{k-1}\hat{f}^{k-1})_{\cdot,\cdot} \\ &\quad -(f^{k-1}_{i,\cdot} - \hat{f}^{k-1}_{i,\cdot}), \epsilon_{k}) \end{aligned}$$

and

$$\begin{aligned} &\tilde{e}^k_{i,j}(1) := \tilde{e}^{k-1}_{i,2j-1},\ \tilde{e}^k_{i,j}(2) := \tilde{e}^k_{2i-1,2j-1}, \\ &\tilde{e}^k_{i,j}(3) := \tilde{e}^k_{2i,2j}, \\ &\hat{e}^k_{i,j}(1) := \hat{e}^{k-1}_{i,2j-1},\ \hat{e}^k_{i,j}(2) := \hat{e}^k_{2i-1,2j-1}, \\ &\hat{e}^k_{i,j}(3) := \hat{e}^{k-1}_{2i,2j}. \end{aligned}$$

Proposition 1 *Given a discrete sequence $\bar{f}^L$, with the modified encoding algorithm for the quincunx cell-average framework in 2-d we obtain a multiresolution representation $M\bar{f}^L = \{\bar{f}^0, \hat{e}^1, \ldots, \hat{e}^L\}$ such that if we apply the decoding algorithm we obtain $\hat{f}^L$ satisfying:*

$$
\begin{aligned}
||\bar{f}^L - \hat{f}^L||_\infty &\leq ||\bar{f}^0 - \hat{f}^0||_\infty & (5)\\
&+ 2\sum_{k=2,\ k=k+2}^{L} |||\tilde{e}^k - \hat{e}^k|||_\infty & (6)\\
||\bar{f}^L - \hat{f}^L||_1 &\leq ||\bar{f}^0 - \hat{f}^0||_1 & (7)\\
&+ \frac{3}{8}\sum_{k=2,\ k=k+2}^{L} |||\tilde{e}^k - \hat{e}^k|||_1 & (8)\\
&+ \frac{5}{8}\sum_{k=2,\ k=k+2}^{L} ||\tilde{e}^k(1) - \hat{e}^k(1)||_1 \\
||\bar{f}^L - \hat{f}^L||_2^2 &= ||\bar{f}^0 - \hat{f}^0||_2^2 & (9)\\
&+ \frac{1}{8}\sum_{k=2,\ k=k+2}^{L} |||\tilde{e}^k - \hat{e}^k|||_2^2 & (10)\\
&+ \frac{7}{8}\sum_{k=2,k=k+2}^{L} ||\tilde{e}^k(1) - \hat{e}^k(1)||_2^2 \\
&+ \sum_{k=2,k=k+2}^{L} |\tilde{E}^k - \hat{E}^k|
\end{aligned}
$$

where

$$
\begin{aligned}
|||\tilde{e}^k - \hat{e}^k|||_\infty &= \max(||\tilde{e}^k(1) - \hat{e}^k(1)||_\infty, \\
&\quad ||\tilde{e}^k(2) - \hat{e}^k(2)||_\infty, \\
&\quad ||\tilde{e}^k(3) - \hat{e}^k(3)||_\infty), \\
|||\tilde{e}^k - \hat{e}^k|||_1 &= ||\tilde{e}^k(1) - \hat{e}^k(1)||_1 + \\
&\quad ||\tilde{e}^k(2) - \hat{e}^k(2)||_1 + \\
&\quad ||\tilde{e}^k(3) - \hat{e}^k(3)||_1, \\
|||\tilde{e}^k - \hat{e}^k|||_2^2 &= ||\tilde{e}^k(1) - \hat{e}^k(1)||_2^2 + \\
&\quad ||\tilde{e}^k(2) - \hat{e}^k(2)||_2^2 + \\
&\quad ||\tilde{e}^k(3) - \hat{e}^k(3)||_2^2, \\
\tilde{E}^k_{i,j} - \hat{E}^k_{i,j} &= \tfrac{1}{4}(2(\tilde{e}^k(1) - \hat{e}^k(1))(\tilde{e}^k(2) - \hat{e}^k(2)) - 2(\tilde{e}^k(1) - \hat{e}^k(1))(\tilde{e}^k(3) - \hat{e}^k(3)) + (\tilde{e}^k(2) - \hat{e}^k(2))(\tilde{e}^k(3) - \hat{e}^k(3))).
\end{aligned}
$$

It is absolutely trivial then to prove the following corollary.

Corollary 1 *Consider the error control multiresolution scheme described in proposition 1, and a processing strategy for the scale coefficients such that for k even*

$$
||\tilde{e}^k(l) - \hat{e}^k(l)||_p \leq \epsilon_k \quad l = 1, 2, 3, \quad p = \infty, 1, \ or\ 2
$$

Then we have

$$
\begin{aligned}
||\bar{f}^L - \hat{f}^L||_\infty &\leq ||\bar{f}^0 - \hat{f}^0||_\infty + 2\sum_{k=2,k=k+2}^{L} \epsilon_k \\
||\bar{f}^L - \hat{f}^L||_1 &\leq ||\bar{f}^0 - \hat{f}^0||_1 + \frac{7}{4}\sum_{k=2,k=k+2}^{L} \epsilon_k \\
||\bar{f}^L - \hat{f}^L||_2^2 &\leq ||\bar{f}^0 - \hat{f}^0||_2^2 + \frac{10}{4}\sum_{k=2,k=k+2}^{L} \epsilon_k^2
\end{aligned}
$$

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