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SVD-Wavelet Algorithm for image compression

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Abstract: We explore the use of the singular value decomposition (SVD) in image compression. We link the SVD and the multiresolution algorithms. In [22] it is derived a multiresolution representation of the SVD decomposition, and in [15] the SVD algorithm and Wavelets are linked, proposing a mixed algorithm which roughly consist on applying firstly a discrete Wavelet transform and secondly the SVD algorithm to each subband. We propose a new algorithm, which is carried out in two main steps. Firstly we decompose the data matrix corresponding to the image following a singular value decomposition. Secondly we apply a Harten's multiresolution decomposition to the singular vectors which are considered significant. We study the compression capabilities of this new algorithm. We also propose a variant of the implementation, where the multiresolution transformation is carried out by blocks. We apply on each block, depending on a selection process, either the algorithm presented or the 2D multiresolution algorithm based on biorthogonal wavelets.

Key Words: Singular value decomposition, multiresolution, image processing.

1 Image compression with the Singular Value Decomposition (SVD)

The SVD matrix decomposition is extensively used in Mathematics. It appears in fields related directly with algebra, such as least squares problems or the calculus of the matrix rank. Its usefulness in applications concerning image processing has also been evaluated. Among these applications we can mention patron recognition, secret communication of digital images, movement estimation, classification, quantization, and compression of images as much as of video sequences (see [9],[10],[14],[16]).

The interest of this transformation comes from the fact that by using it we obtain easily the best approximation of a given rank in terms of the 2-norm and the Frobenius norm.

The fundamental results are collected in the following theorems.

Theorem 1 Singular Value Decomposition of a matrix *([18])*
Given a real matrix A of size $n \times m$ and rank r, there exist two matrices U and V of sizes $n \times n$ and $m \times m$ respectively such that

$$A = U \begin{pmatrix} \sum & 0 \\ 0 & 0 \end{pmatrix} V^t \tag{1}$$

where $\sum = diag(\sigma_1, \sigma_2, \ldots, \sigma_r)$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0$.

The values $\{\sigma_i\}_{i=1}^r$ are called singular values of the matrix A and they coincide with the positive square roots of the eigenvalues of the matrix AA^t and of A^tA. Operating this matrix product we get

$$A = \sum_{i=1}^{r} \sigma_i U_i V_i^t \tag{2}$$

where U_i and V_i denote the column vectors of the matrices U and V respectively. They are called singular vectors associated to the singular value σ_i.

Theorem 2 Approximated rank *([18])*
We consider the SVD decomposition of a matrix A of size $n \times m$ as in Theorem 1. If $k < r = rang(A)$ and

$$A_k = \sum_{i=1}^{k} \sigma_i U_i V_i^t \tag{3}$$

then:

$$\begin{aligned} min_{rang(B) \leq k} ||A - B||_2 &= ||A - A_k||_2 \\ &= \sigma_{k+1} \end{aligned}$$

$$\begin{aligned} min_{rang(B) \leq k} ||A - B||_F^2 &= ||A - A_k||_F^2 \\ &= \sum_{i=k+1}^{r} \sigma_i^2 \end{aligned}$$

This theorem tells us how to obtain approximations of small rank to a matrix. Besides, these approximations are the best ones in terms of the 2-norm and the Frobenius norm among all the possible approximations with equal or smaller rank. This result has been used to compress an image in the following way. An image (matrix) A of size $n \times m$ has initially $n \times m$ entries to store. If we consider A_k, defined as

$$A_k = \sum_{i=1}^{k} \sigma_i U_i V_i^t \tag{4}$$

instead of A, then we have an approximation of A which can be stored with $k(n + m + 1)$ values, ie., the entries of the vectors U_i, V_i and the singular values σ_i, $i = 1, \ldots, k$. Clearly, a compromise between the precision of the approximation and the desired compression ratio must be achieved. The compression algorithm is competitive when with a small value of k we get already a good quality of the resulting image.

2 A SVD-Wavelet method

Multiscale transformations can help to improve the compression capabilities of the SVD algorithm. The SVD algorithm described in the previous section can be optimized in various ways by incorporating further compression mechanisms in the vectors that appear in the SVD decomposition of the image.

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Let us consider an index k such that

$$A_k = \sum_{i=1}^{k} \sigma_i U_i V_i^t \tag{5}$$

gives the desired image quality. We can expect to obtain greater compression by applying a multiresolution transformation (MR henceforth) to the one dimensional vectors U_i, V_i that appear in the expression for A_k.

The MR algorithm used for this purpose can be chosen by the user. We use Harten's framework because of its simplicity both in theoretical analysis and in practical implementation. In this paper we shall implement the order three 1D cell average MR algorithm in [8]. We recall it in detail for the sake of completeness.

2.1 1D multiresolution algorithm for cell averages

Algorithm 1 *Coding for cell averages*

$$\begin{array}{l}
\text{for } k = L, \ldots, 1 \\
\quad \text{for } j = 1, \ldots, J_{k-1} \\
\qquad \bar{f}_j^{k-1} = \frac{1}{2}(\bar{f}_{2j-1}^k + \bar{f}_{2j}^k) \\
\quad \text{end} \\
\quad \text{for } j = 1, \ldots, J_{k-1} \\
\qquad \hat{d}_j^k = \bar{f}_{2j-1}^k - (P_{k-1}^k \bar{f}^{k-1})_{2j-1} \\
\quad \text{end} \\
\text{end} \\
M\bar{f}^L = \{\bar{f}^0, \hat{d}^1, \ldots, \hat{d}^L\}
\end{array}$$

Algorithm 2 *Decoding for cell averages*

$$\begin{array}{l}
\text{for } k = 1, \ldots, L \\
\quad \text{for } j = 1, \ldots, J_{k-1} \\
\qquad \bar{f}_{2j-1}^k = (P_{k-1}^k \bar{f}^{k-1})_{2j-1} + \hat{d}_j^k \\
\qquad \bar{f}_{2j}^k = 2\bar{f}_j^{k-1} - \bar{f}_{2j-1}^k \\
\quad \text{end} \\
\text{end} \\
M^{-1}\{\bar{f}^0, \hat{d}^1, \ldots, \hat{d}^L\} = \bar{f}^L
\end{array}$$

3 A Block SVD-Wavelet algorithm

To obtain a better adaptation to the concrete characteristics of a given image we apply the SVD-Wavelet algorithm to the matrix by blocks. In this way we expect some advantages, since if an area of the image is quite simple then we will store it using only a few singular values. We consider blocks 64×64 and 3 levels of multiresolution.

3.1 Selection of the blocks in which we apply the SVD-Wavelet algorithm

The selection consist in applying the SVD-Wavelet algorithm to one block if the number of singular values needed to maintain the algorithm below a certain level of error is not bigger than an integer λ_s, whose value depends on the size $s \times s$ of the blocks. If that requisite is not satisfied, then we apply instead the 2D multiresolution algorithm to that block. We choose a linear centered reconstruction operator of the same order, in our tests third order.

3.2 Numerical Experiments

We apply the SVD and SVD-Wavelet method to some tests images with the purpose of studying its pros and cons. We compare these algorithms with the 2D MR algorithm based on tensor product.

The $PSNR$ (Peak Signal to Noise Ratio) given by

$$PSNR = 20 \log_{10} \frac{255}{\sqrt{MSE}}, \tag{6}$$

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where MSE means the mean square error

$$MSE = \frac{||A - A_k||_F^2}{nm}, \tag{7}$$

and the number 255 is due to the 8 bit gray scale, gives an indication of the quality of the reconstructed image. The larger the PSNR, the better the approximation.

We define the compression ratio as

$$r_c = \frac{\text{entries of the original image}}{\text{entries stored in the compressed version}} \tag{8}$$

The results appear in Tables 1-2-3.

SVD	k vs	r_c	$PSNR$
squares	4	31.94	296.28
circles	50	2.56	31.98
circles	100	1.28	43.41
circles	150	0.85	293.21
geome	35	7.31	36.16
geome	40	6.39	37.20
geome	50	5.12	39.06
geome	100	2.56	46.58
geome	150	1.71	53.47
geome	200	1.28	62.44
tiffany	50	5.12	33.75
tiffany	100	2.56	37.46
tiffany	150	1.71	40.36
tiffany	200	1.28	43.19
lena	50	5.12	30.18
lena	100	2.56	35.65
lena	150	1.71	39.87
lena	200	1.28	43.24

Table 1: Image quality and compression ratios for the SVD, Algorithm using k singular values.

$SVD - Wav$	k vs	tol	r_c	$PSNR$
squares	4	10	158.30	51.68
geome	50	10	32.13	36.61
geome	100	10	19.96	37.34
tiffany	50	10	27.68	32.23
tiffany	100	10	16.03	32.87
tiffany	150	10	12.48	32.91
tiffany	200	10	10.62	32.92
lena	50	10	18.84	29.47
lena	100	10	10.29	31.94
lena	150	10	8.20	32.20
lena	200	10	7.30	32.21

Table 2: Image quality and compression ratios for the SVD-Wavelet method using k singular values and $tol = 10$ in the multiresolution algorithm.

$Wav2D$	tol	r_c	$PSNR$
squares	5	33.59	59.92
squares	10	60.96	44.63
geome	5	30.51	50.21
geome	10	45.19	43.18
tiffany	5	12.50	38.81
tiffany	10	29.53	35.41
lena	5	9.56	37.99
lena	10	19.72	34.65

Table 3: Image quality and compression ratios for the 2D linear multiresolution algorithm: biorthogonal wavelets (via tensor product) which uses a linear reconstruction of order 3 based on Lagrange interpolation.

4 Conclusions and perspectives

We have presented a method that allows to modify the singular vectors of the SVD in such a way that a larger compression is attained. We have linked the SVD algorithm with Harten's multiresolution framework. We have applied this method to compress images, and we have seen that the lower the rank of the original matrix and the smaller the singular values, the better the results. We are led to say that we get very good results when the SVD method was already competitive, improving it in terms of compression ratios.

A possibility to improve the method could be to modify certain entries of the matrix in such a way that visually the image is not highly affected, and that with this modification the new matrix has smaller rank. This would lead us to the theory of 'matrix completeness'. Anyway as already said the method is not competitive for an arbitrary image, and we think it is not worth to follow in this line of research if the objective is just image compression.

However there exist other applications in which the SVD is useful and is being applied nowadays. Among such applications are the steganography (secrete communication of images immersed inside others) or the watermarking (copyright protection in digital products). Therefore, we think it is interesting the fact of being able to modify the singular vectors of the SVD at the same time that we maintain the error committed under control. This is a future subject of research [17].

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