Fractal Robustness and Parameter Tuning PI^αD^β Controllers

ABDELBAKI DJOUAMBI*, ABDELFATAH CHAREF** and TAREK BOUKTIR^*

* University Larbi Ben M’Hidi, Oum El-Bouaghi, Algeria
** University Mentouri, Ain El Bey, Constantine, Algeria

Abstract: This paper presents a transposition of fractal robustness in automatics through the fractional order control and fractional feedback control of some of typical systems. Fractal robustness expresses the robustness of damping in nature, that fractality ensures through non integer derivation. This concept is illustrated by the relaxation of water on a porous dyke, its damping being independent of the motion of water mass. This robust phenomenon is paradoxical in the integer approach of mechanics, where any relaxation presents a damping linked to the carried mass. The considered dynamic model which governs this phenomenon is a non integer order linear differential equation where the natural frequency and the damping ration of the oscillatory mode of the solution are determined. Using such model as an open-loop reference model, a new tuning strategy of non integer order PI^αD^β controller has been presented for some typical systems. The performances of robustness is compared and illustrated with an ordinary PID controller through frequency and time responses.

Key-words: fractal systems, fractional feedback control, tuning, robustness.

1 Introduction

As summarized in [1], [2] and [3], many real-world physical systems are well characterized by fractional order differential equations. In particular, it has been shown that the relaxation of water on a porous dyke can be more adequately molded by fractional order integrator [4]. The analyses seems to show that the relaxation is characterized by a natural frequency which depends on the motion water mass and by a damping ration which is independent of it.

Although it appears paradoxical in the integer approach of mechanics, where any relaxation presents a damping linked to the carried mass, this result reveals the insensitivity of the damping ration to a parameter at least. It therefore expresses the robustness of the stability degree of the relaxation phenomenon.

The aim of the following development consists in the transposition of such robustness in control systems through fractional order PI^αD^β controller using an elementary fractional order integrator as a reference model of the open-loop control system.

2 Fractal robustness

The concept of fractal robustness may be illustrated by the relaxation of water on a porous dyke, where the damping ration is independent of the mass of moving.

Given a mass M of water and the flow section S, considering the fractal character of the dyke porosity and the corresponding recursivity, it can be demonstrated that the pressure P(t) at the water-dyke interface has the following differential equation [3]:

\[ \frac{M}{S^2 w_0^{-1}} \left( \frac{d}{dt} \right)^\alpha P(t) + P(t) = 0 \, , \, \, 1<\alpha<2 \]  

(1)

This equation represents the dynamic model of the water dyke interface.

Taking the Laplace transform of equation (1), we obtain:

\[ (s^\alpha + 1) P(s) + P(s) = 0 \]  

(2)

Where

\[ \tau = \left( \frac{M}{S^2 w_0^{-1}} \right)^{1/\alpha} \]  

(3)

This operational equation is translated by the functional diagram of Figure 1 which reminds that of a free control loop \( (E(s)=0) \). Because of a unit feedback, the direct chain determines an open loop transmittance of the form:

\[ \beta(s) = \left( \frac{1}{\tau s} \right)^\alpha = \left( \frac{w_0}{s} \right)^\alpha \]  

(4)

which is just the transmittance of an non integer integrator whose unit gain frequency is \( w_0 = 1/\tau \).

Fig.1 Functional diagram making it possible to define an open loop transfer.
Given that \( \arg \beta(jw) = -\alpha \frac{\pi}{2} \) with \( 1 < \alpha < 2 \), the Nichols locus of \( \beta(jw) \) is a vertical straight line abscissa between \(-\pi/2\) and \(-\pi\).

When water mass \( M \) changes, frequency \( w_u \) is modified in conformity with the relation

\[
 w_u = \left( w_0^{\frac{1 - \alpha}{\alpha}} \frac{S^2}{M} \right)^{\frac{1}{\alpha}}
\]  

(5)

So, the template slides on itself at the time of variation of the water mass. Such a vertical displacement of a variation of the template ensures the constancy of phase margin \( \phi_m \) (Figure 2). This expresses the robustness of stability degree.

**Fig. 2** Illustration of fractal robustness.

In controller design, the objective is to achieve such a similar frequency behavior, in a medium frequency range around \( w_u \), knowing that the closed loop dynamic behavior is exclusively linked to the open loop behavior around \( w_u \). Therefore, the ideal controller design comprises:
- an open loop Nichols locus which forms a vertical straight line segment around \( w_u \) for the nominal parametric state of plant,
- and a sliding of template on itself when there exist parameter change in plant.

The search for the synthesis of such a template defines the non integer approach that the second generation CRONE control uses [5].

### 3 Elementary fractional Order control

In this section, we present the fundamental characteristics of an elemental control system with an open-loop transfer function given by a fractional order integrator. Based on this study, in the next section, it is developed a tuning method for \( P^\alpha D^\beta \) controller for some of typical plants.

Let us consider the unit feedback system represented in Figure 3 with open-loop transfer function

\[
 G(s) = \frac{K}{s^m}, \quad s = jw, \quad (1 < m < 2)
\]  

(6)

#### 3.1 Frequency domain characteristics

The open-loop Bode diagrams of amplitude and phase shown in Figure 4 have a slope of \(-20 m \text{dB/dec}\) and a constant phase of \(-m\pi/2\) rad. Therefore, the closed-loop system has a constant phase margin of:

\[
 \phi_m = (1 - m)\pi \quad \text{rad}
\]  

(7)

that is independent of the gain \( K \).

**Fig. 3** Elemental control system with fractional order integrator \((1 < m < 2)\).

**Fig. 4** Open-loop Bode diagram of \( G(s) \), \((1 < m < 2)\).

Consider now \( H(s) \), the closed-loop transfer function of the unit feedback system presented in Figure 3:

\[
 H(s) = \frac{Y(s)}{E(s)} = \frac{G(s)}{1 + G(s)} = \frac{1}{(s/w_u)^m + 1}
\]  

(8)

where \( w_u = k^{1/m} \) denotes the open-loop crossover frequency. The asymptotic approximation of equation (8) indicates that the magnitude (phase) asymptotically approaches a horizontal straight line at 0 dB and (0 rad) as \( w/w_u \to 0 \) and a straight line of \(-20\text{dB/dec} \) \((- m\pi/2\) rad) as \( w/w_u \to \infty \).
The resonance peak $M_r$ and the frequency at which occurs $w_r$ are given by the formulae [6]:

$$M_r = \frac{1}{\sin(m\pi/2)}$$  \hspace{1cm} (9)

$$w_r = w_u \left| \cos(m\pi/2) \right|^{1/m}$$  \hspace{1cm} (10)

### 3.2 Time domain characteristics

Fractional order systems have an infinite dimension [7]. Proper approximation by finite difference equation is needed. Specifications for a control system design often involve certain requirement associated with the time response. Hence, for the purpose of time domain analysis, synthesis and simulations, the need arises for a rational function approximation [8] and [9].

The asymptotic behavior with slope of -$20m$ dB/dec of the function (6) is approximated by a number of zig-zag lines connected together with alternate slopes as represented in Figure 5:

![Fig 5 Broken-line approximation of $G(s)$](image)

where $\varepsilon$ denotes the approximation error,

$$\rho = \frac{2\varepsilon}{(1-m)m}, \quad \alpha = \frac{\varepsilon}{m}$$  \hspace{1cm} (11)

Given a frequency band of interest $[w_{\text{min}}, w_{\text{max}}]$, the approximated function of $G(s)$ can be expressed by the following equation:

$$G(jw) = \frac{1}{(w/w_u)^m} \approx \frac{1}{w_{\text{min}}^m} \prod_{i=1}^{N} (1 + jw/w_{i1})$$  \hspace{1cm} (12)

where $w_{i1} = w_{\text{min}}10^\alpha$ and $w_{i2} = w_{i1}10^\alpha$.

for $i \geq 2$

$$w_{i1} = 10^\alpha w_{i-1,1}, \quad w_{i2} = w_{i1}10^\alpha.$$  \hspace{1cm} (13)

Given an approximation error $\varepsilon$, the number of cells $N$ is obtained by the following equation:

$$N = E \left\{ \frac{m(1-m)}{2\varepsilon} \left[ \log_{10}(w_{\text{max}}) - \log_{10}(w_{\text{min}}) \right] \right\} + 1, \quad (14)$$

Using equation (8) and equation (12) the closed loop approximated transfer function is presented by the following equation:

$$H(jw) \approx \prod_{i=1}^{N} (1 + jw/w_{i1})$$

Figures 6 and 7 present the frequency response of the ideal function and that of its approximation with $N=10$, $m=1.5$ and frequency band $[10^{-6}, 10^6]$. The figures show that the two curves are overlapping. Taking the Laplace inverse transform of equation (15), the time responses of the considered closed loop system for $m=(1.1, 1.5, 1.9)$ and $N=10$ is presented in figure 8.

Figure 9 shows the step responses of the closed loop system for different values of the system gain $k$ with $m=1.5$. It can be seen that the first overshoot remains constant, showing a robustness that characterizes the considered fractional system.
The percent overshoot $M_p(\%)$ versus $m$ (Figure 10) can be approximated by the expressions:

$$M_p = \frac{h_{\text{max}} - h(\infty)}{h(\infty)}$$  \hspace{1cm} (16)$$
$$M_p \approx 0.8(m - 1)(m - 0.75), \hspace{1cm} 1 < m < 2 \hspace{1cm} (17)$$

The peak time $T_p$ (Fig.11) and the rise time $T_r$ (Fig.12) of the unit step response are given by the approximate formulas:

$$w_u T_p \approx \frac{1.1057(m - 0.2546)^2}{(m - 0.9205)}, \hspace{1cm} 1 < m < 2 \hspace{1cm} (18)$$
$$w_u T_r \approx \frac{0.1306(m + 1.1607)^2}{(m - 0.7035)}, \hspace{1cm} 1 < m < 2 \hspace{1cm} (19)$$

4 Tuning method of PI$^D$D$^\mu$ controllers

We have established a simple design of an elementary fractional order control system based on the parameters $m$ and $w_u$. Therefore, in this section we address the Fractional Order Integrator of equation (6) as reference function $G_m(s)$ for PI$^D$D$^\mu$ control system. We start by considering the closed-loop system shown in fig.13, where $C(s)$ is the PI$^D$D$^\mu$ controller and $G_p(s)$ the plant transfer function characterized by an asymptotic order at low frequency $0 \leq n' \leq 2$ and at high frequency $2 \leq n \leq 4$ with $n' < n$ (figure 14).
are positive real numbers, $-\pi$, (24)

The ideal transfer function of a closed-loop performances.

interpretation of the open-loop transfer function approximation of the open loop transfer function following form:

Then we can write:

which can be written as:

The transmittance superior than 10 times the transitional frequency of the plant.

where, $w_o$ denotes the transitional frequency of the plant, the transmittance $T(s)$ should presents:

- an asymptotic slope of $-20m$ dB/dec at low and high frequency of the limited bandwidth $[\omega_{min},\omega_{max}]$ which allows to calculate the parameters $\lambda$ and $\mu$.
- the same magnitude with the reference $G_m(s)$ in the high and low frequency, so, the initial values of parameters $T_i'$ and $T_d'$ can be estimated with an initial value of $K_p$ considered to be $K_p = 1$.
- a crossover frequency equal to $w_o$, the parameter $K_p$ can be deduced using the initial values $T_i'$ and $T_d'$.
- finally: adjusting the parameters $T_i'$ an $T_d'$ to be:

Taking $n$ and $n'$ as the asymptotic order of the plant $G_p(s)$ at high an low frequency respectively, the controller parameters can be given by the following equations:

$$\mu = n - m, \quad \lambda = m - n', \quad (24)$$

$$T_d = \frac{K_u}{G_p(j\omega_{max})w_{max}^n}, \quad (25)$$

$$T_i' = \frac{K_u}{G_p(j\omega_{min})w_{min}^n}, \quad (26)$$

$$K_p = \left| \frac{G_p(jw_o)}{1 + T_i'(jw_o)^{\lambda} + T_d'(jw_o)^{\mu}} \right|, \quad (27)$$

$$T_i' = T_iK_p \quad \text{and} \quad T_d' = T_dK_p \quad (28)$$

5 Simulation results

In these section we analyze the open-loop frequency and closed-loop time response characteristics of the $\mathcal{P}^2D^\mu$ system tuned according to the reference design parameters $(m, \omega_0)$. In order to establish different case studies, we adopt three distinct processes, given by:

$$G_{p1} = \frac{g_0}{(s/w_0)^2 + 2\epsilon(s/w_0) + 1} \quad (39)$$

$$G_{p2} = \frac{g_0}{s(1 + \frac{s}{w_1})w_0} \quad (30)$$

$$G_{p3} = \frac{g_0}{s(1 + \frac{s}{w_0})^2} \quad (31)$$
Given a frequency band of interest: $[10^{-6}, 10^0]$ rad/sec and the parameter values of the above plants as follow:

- plant $G_{p1}(s)$: $g_0 = 10$, $w_0 = 33.33$ rd/s and $\varepsilon = 0.01$.
- plant $G_{p2}(s)$: $g_0 = 1$, $w_0 = 50$ rd/s and $w_1 = 16.89$ rd/s.
- plant $G_{p3}(s)$: $g_0 = 1$, $w_0 = 10$ rd/s and $w_1 = 17$ rd/s.

From the desired closed-loop performances we can fix the order $m$ and the crossover $w_u$ of the reference model (Sec. 3).

The reference model and $P\hat{D}D^\v$ closed-loop system are simulated in time domain by use of equation (12) with $N=10$.

In order to check the achieved approximation in frequency domain we present in Fig. 15, 16, 17, 18, 19 and 20 the Bode diagrams of amplitudes and phases for the reference model $G_m(s)$ and $P\hat{D}D^\v$ open-loop systems of $G_{p1}(s)$, $G_{p2}(s)$ and $G_{p3}(s)$ with the reference model design parameters $\phi_m = 45^\circ$ and $w_u = 500$ rad/s. This tuning corresponds to $m=1.5$. We see that the curves are very similar in both cases.

Figures 21, 22 and 23 present the step response of the reference model and the $P\hat{D}D^\v$ closed-loop systems with the previous case. The figures show that the curves are very similar.
6 Performances

In this section we present a comparative study of the robustness performances obtained with the $P^D\theta$ controller and a classical type PID controller. The plant is a second order system given by the following transfer function:

$$G(s) = \frac{g_0}{s^2 + 2\varepsilon s + \omega_0^2}$$  \hspace{1cm} (32)

where $g_0$ denotes the static gain, $\omega_0$ the transitional frequency and $\varepsilon$ the damping ration.

The nominal values of the parameters $g_0$, $\omega_0$ and $\varepsilon$ are given as follow:

$$g_0 = 10, \quad \omega_0 = 33.33, \quad \varepsilon = 0.01. \hspace{1cm} (33)$$

In order to achieve the robustness propriety of each controller, the values of $g_0$ and $\omega_0$ are modified as follow:

$$
\begin{align*}
    g_{0\text{ min}} &= \frac{g_0}{10} ; \quad g_{0\text{ max}} = 10g_0 , \\
    \omega_{0\text{ min}} &= \frac{\omega_0}{2} ; \quad \omega_{0\text{ max}} = 2\omega_0 .
\end{align*} \hspace{1cm} (34)
$$

The fractional controller $P^D\theta$ and classical type PID are calculated for the considering plant under the nominal values of equation (33) and the desired closed-loop frequencies specifications given as follow:

$$\phi_{\text{in}} = 45^\circ, \quad w_{\text{u}} = 500 . \hspace{1cm} (35)$$

The transfer function of a classical PID controller has the following form:

$$c_c(s) = c_0 \frac{(1+s/z_1)(1+s/z_2)}{(1+s/p_1)(1+s/p_2)}$$  \hspace{1cm} (36)

in which:

$$c_0 = 50.797 ; \quad z_1 = 3.7796 \text{ rad/s} ; \quad z_2 = 188.982 \text{ rad/s} ; \quad p_1 = 0.6299 \text{ rad/s} ; \quad p_2 = 1322.90 \text{ rad/s} .$$

The transfer function of a fractional $P^D\theta$ controller has the following form:

$$C(s) = K_p \left(1 + \frac{T_s}{s^{\lambda}} + T_d s^{\mu}\right)$$  \hspace{1cm} (37)

using equations (24,...,28) :

$$
\begin{align*}
    K_p &= 0.97 ; \quad T_s = 1083.30 ; \quad T_d = 0.9752 , \\
    \lambda &= 1.5 ; \quad \mu = 0.5 .
\end{align*}
$$

As shown in Figure 24 and figure 25, fractional order $P^D\theta$ controller realizes better robustness versus static gain plant variation with smaller overshoot and fastest rise time. Figure 26 gives the step responses of the $P^D\theta$ control system with different values of $w_{0\text{ min}}$ compared to the step responses of the classical type PID control system (Figure 27) with the same variation of $w_{0\text{ min}}$, it can be shown that fractional order $P^D\theta$ controller realizes better robustness against plants parameters variation.
of damping is illustrated by a frequency template in the Nichol’s plane whose form and vertical sliding ensure the invariance of the phase margin.

An intuitive broken-line approximate synthesis method of frequency-band controllers is also introduced which has a satisfactory accuracy in frequency domain.

The experimental results with several typically plants show the robustness of proposed fractional $P^\mu D^\nu$ control system compared to a classical PID controller.

8 References

7 Conclusion
In this paper, an alternative point of view for the tuning of robust fractional $P^\mu D^\nu$ controller has been presented. The synthesis method is based on the interpretation of the open loop $P^\mu D^\nu$ control system witch can be considered to be of an elementary fractional order integrator model. Such model governs the relaxation of water on porous dyke, it consists in a differential equation of non integer order between 1 and 2. Damping is indeed specific to the non integer of differential equation imposed by the fractal dimension of the dyke. This expresses a remarkable property: fractality determines damping in nature. The robustness