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Nonuniform DFT based on nonequispaced sampling

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Abstract: - In this paper, we propose a nonuniform DFT based on nonequispaced sampling in the frequency domain. It is useful to detect some specific frequencies such as in DTMF which is composed on two different frequencies, main (fundamental) frequency component among lots of harmonics, and feature detection from noisy signals. Some trials for nonuniform data processing via DFT are discussed and resampling method to obtain nonlinearly spaced samples in the frequency domain.

Key-Words: - NDFT, DFT, Resampling, HT, WDFT

1 Introduction

For signals consisting of a number of frequency components, the Fourier Transform (FT) effectively reveals their frequency contents and is generally able to represent the signals with an acceptable resolution divided by equal bandwidth in the frequency domain. The discrete Fourier transform (DFT) is an important tool in digital signal processing. The N-point DFT of a length-N sequence is given by the frequency samples of the z-transform at N-uniformly spaced points [1]. The nonuniform DFT (NDFT) proposed recently [2] is the most general form of DFT that can be employed to evaluate the frequency samples at N arbitrary but distinct points in the z-plane. Since the unitary property is not inherently guaranteed, some fast computation algorithms have been designed by using the approximation algorithm [3-5]. Besides for developing the enhanced analysis of nonuniform data, the idea is applied for detecting harmonics related to the fundamental frequency [6] and the input data is warped prior to the DFT to provide nonuniform frequency spacing [7].

However the main problem in this nonuniform processing is to define the nonuniform sampling in the time or frequency domain. It must be adaptable to the signal property and the most featured components. In this paper we proposed resampling method in the frequency domain to obtain nonequispaced transformation. DFT coefficients can be the first criteria to approximate the main components. Nonuniform sampling is achieved in the DFT domain by interpolation using spline method. Final resampling is achieved by rearranging the interpolated samples.

2 Motivation of DFT and NDFT

2.1 DFT

Let us take into consideration the definition of Fourier transform in the continuous domain first: Under certain conditions upon the function $s(t)$, the Fourier transform of this function exists and can be defined as

$$S(w) = \int_{-\infty}^{\infty} s(t) e^{-jwt} dt \qquad (1)$$

where $w = 2\pi f$ and f is a temporal frequency. The original signal is recovered by the inverse Fourier transform (IFT), given by:

$$s(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(w) e^{jwt} dw \qquad (2)$$

From the definition of FT, we consider now the discrete Fourier transform. In this case we have a finite number N of samples of the signal $s(t)$ taken at regular intervals of duration T_s (which can be considered a sampling interval). In practical cases the signal $s(t)$ has not an infinite duration, but its total duration is $T = NT_s$ and we have a set $\{s_n\}$ of samples of the signal $s(t)$ taken at regular intervals. We can define $s_n \doteq s(t_n)$, where $t_n = nT_s$, for $n = 0,1,...,N-1$, is the sampling coordinate.

In the case of the discrete Fourier transform, not only we want the signal to be discrete and not continuous, but we also want the Fourier transform, which is a function of the temporal frequency, to be defined only at regular points of the frequency domain. Thus the function $S(w)$ is not defined for every value of w but only for certain values w_m. We want the samples $S(w_m)$ to be regularly spaced as well, so that all the samples w_m are multiples of a dominant frequency $\frac{1}{T}$, that is to say $w_m = m\left(\frac{2\pi}{T}\right)$, for $m = 0,1,...,N-1$. Let us note now that T is equal to the finite duration of the signal $s(t)$ from which we want to define its discrete Fourier transform. Note also that we assume the number of samples in frequency to be equal to the number of samples in the temporal domain, that is N. This is not a necessary condition, but it simplifies the notation.

The direct extension of Eq. (1) to the discrete domain is:

$$S(w_m) = \sum_{n=0}^{N-1} s(t_n) e^{-jw_m t_n} \qquad (3)$$

Considering that w_m is defined as only discrete values $m\left(\frac{2\pi}{T}\right)$ and t_n is also defined as only discrete values nT_s, it is possible to rewrite Eq. (3) as:

$$S(w_m) = \sum_{n=0}^{N-1} s(t_n) e^{-j\left(m\frac{2\pi}{T}\right)(nT_s)} = \sum_{n=0}^{N-1} s(t_n) e^{-j\left(m\frac{2\pi}{NT_s}\right)(nT_s)} \qquad (4)$$

It is now possible to simplify and express the dependence on w_m only in terms of m and the dependence on t_n only in terms of n. In that way the final definition of the DFT is:

$$S(m) = \sum_{n=0}^{N-1} s_n\, e^{-j\frac{2\pi}{N}mn} = \sum_{n=0}^{N-1} s_n\, W_N^{mn}, \quad where\ W_N = e^{-j\frac{2\pi}{N}} \qquad (5)$$

And the inverse of the discrete Fourier transform (IDFT) as:

$$s_n = \sum_{m=0}^{N-1} S(m) e^{j\frac{2\pi}{N}nm} = \sum_{m=0}^{N-1} S(m) W_N^{-nm} \qquad (6)$$

2.2 NDFT

Now we want to generalize the definition and the computation of the Fourier transform from the regular sampling to the irregular sampling domain. In the general case, the definition of the Nonuniform Discrete Fourier Transform is the same as the one given by Eq. (3), taking into consideration that the samples can be taken at irregular intervals both in time (t_n) and/or in frequency (w_m).

However, in practice, we want to take into consideration a more restricted case, which is the case where the samples are irregularly taken in the time domain t but regularly taken in the frequency domain. That is to say that the samples $s(t)$ of the irregular Fourier transform are taken at multiples of a quantity Δk, which is a fixed quantity in the Fourier domain. The fixed quantity Δk in the regular case corresponds to $\frac{2\pi}{T}$. The extension from regular to irregular sampling, therefore, depends on the duration of the signal $s(t)$ and not on the fact that the samples t_n are taken at regular or irregular intervals.

The definition of the nonuniform discrete Fourier transform is as follows:

$$S(m) = \sum_{n=0}^{N-1} s_n\, e^{-jm\Delta k\, t_n} \qquad (7)$$

It is common practice to set $\Delta k = \frac{2\pi}{T}$ where T is the range of extension for the samples t_n. In that case the formulation of the NDFT is very similar to the one of the DFT except of the presence of the spatial coordinates t_n instead of the index n. In this case, the NDFT is defined as:

$$S(m) = \sum_{n=0}^{N-1} s_n\, e^{-j\frac{2\pi}{T}m\, t_n} \qquad (8)$$

From a computational point of view, two differences have to be noticed between DFT and NDFT. The first difference is that samples in frequency are taken at intervals $\frac{2\pi}{T}$ in the irregular case instead of $\frac{2\pi}{N}$ in the regular case (T being the duration of the signal $s(t)$, with $t \in [0,T]$, and N is the number of samples of the signal $s(t)$). The second difference is that, instead of the integer index n in the regular case, in the irregular case the irregular sampling coordinate t_n appears in the exponent. However in a practical view point, the time duration T should be replaced with the number of samples N in the discrete case. Thus we assume the difference between NDFT and DFT is the time interval among samples in the time domain.

2.3 HT

Based on the fact that the time-varying harmonic signal generally contains the fundamental and a number of harmonics, the harmonic transform (HT) is defined by [6]

$$S_{\phi_u(t)}(w) = \int_{-\infty}^{+\infty} s(t)\phi_u^{'}(t)e^{-jw\phi_u(t)}dt \qquad (9)$$

where $\phi_u(t)$ is the unit phase function of the fundamental divided by its nominal instantaneous frequency and $\phi_u^{'}(t)$ is the first-order derivative. We assume a signal $s_h(t)$ consisting of the fundamental and harmonics as

$$s_h(t) = \sum_{k=0}^{\infty} a_k e^{j(k+1)\alpha(t)} \qquad (10)$$

where a_k is the amplitude of the k th harmonic and $\alpha(t)$ is the phase function of the fundamental. Note that the derivative of the phase function is equivalent to the instantaneous frequency, i.e., $\alpha^{'}(t) = c_0(t)$. If $\alpha(t) = \phi_u(t)$ in Eq. (9), the HT of $s_h(t)$ is

$$\begin{aligned} S_{\alpha(t)}(w) &= \int_{-\infty}^{+\infty} \sum_{k=0}^{\infty} a_k e^{j(k+1)\alpha(t)}\alpha^{'}(t)e^{-jw\alpha(t)}dt \\ &= \int_{-\infty}^{+\infty} \sum_{k=0}^{\infty} a_k e^{j(k+1)\alpha(t)} \frac{d\alpha(t)}{dt} e^{-jw\alpha(t)}dt \\ &= \sum_{k=0}^{\infty} a_k \int_{-\infty}^{+\infty} e^{j(k+1)\alpha(t)}e^{-jw\alpha(t)}d\alpha(t) \qquad (11) \\ &= \sum_{k=0}^{\infty} a_k F\left[e^{j(k+1)\alpha(t)}\right] \\ &= \sum_{k=0}^{\infty} 2\pi a_k \delta(w-k-1) \end{aligned}$$

which is an impulse-train for arbitrary $c_0(t)$ frequency. Thus, the HT can provide an impulse-train spectrum instead of the all-pass spectrum in the DFT.

The phase function in the HT is not uniformly defined, but should be available before calculating the HT. Hence, it is a kind of nonuniform DFT and the method to define the phase function is absolutely important.

2.3 WDFT

The warped DFT (WDFT) is a special case of the general NDFT. As suggested by Makur and Mitra, the N-point WDFT $S(m)$ of a length- N sequence s_n is given by N equally spaced frequency samples of a modified z-transform by applying some transforming technique, e.g., an M -th order real coefficient allpass function [7].

The simplest example of a nontrivial mapping is obtained using a first-order allpass function having the zero coefficients, $A(\hat{z}) = -a + \hat{z}^{-1}$ and the corresponding mirror-image polynomial of $A(\hat{z})$, $\tilde{A}(\hat{z}) = 1 - a\hat{z}^{-1}$ for the pole coefficients. That is, the allpass function is

$$z^{-1} = \frac{-a + \hat{z}^{-1}}{1 - a\hat{z}^{-1}} \qquad (12)$$

where $|a| < 1$ for stability. Replacing $z = e^{jw}$ and $\hat{z} = e^{j\hat{w}}$ to define frequency warping on the unit circle, where the original angular frequency is w and the warped frequency is $\hat{w}$, Eq. (12) is rewritten as

$$e^{-jw} = \frac{-a + e^{-j\hat{w}}}{1 - ae^{-j\hat{w}}} = \frac{(-a + e^{-j\hat{w}})(1 - ae^{j\hat{w}})}{(1 - ae^{-j\hat{w}})(1 - ae^{j\hat{w}})} = \frac{(e^{-j\hat{w}/2} - ae^{j\hat{w}/2})^2}{|1 - ae^{-j\hat{w}}|^2} \qquad (13)$$

Taking the square root of each side, we get

$$e^{-jw/2} = \frac{(e^{-j\hat{w}/2} - ae^{j\hat{w}/2})}{|1 - ae^{-j\hat{w}}|} \qquad (14)$$

Again taking the ratio of the imaginary part to the real part of each side, the frequency mapping is obtained to be

$$\tan\left(\frac{w}{2}\right) = \left(\frac{1+a}{1-a}\right)\tan\left(\frac{\hat{w}}{2}\right) \qquad (15)$$

Thus, the original frequency spacing w and the warped one $\hat{w}$ are not linearly dependent, revealing that the allpass transformation warps the frequency scale and uniformly spaced points on the unit circle are mapped onto nonuniformly spaced points on the unit circle in the z-plane.

However the WDFT has some drawbacks. First, the NDFT result is obtained by an extra allpass filter, not by the pure NDFT technique. Second, we still have problem to define the frequency to be warped on the unit-circle. Besides, it is need to transform the frequency points out of the unit-circle.

3 Realization of NDFT

3.1 Forward NDFT

The DFT can be defined as a subset of the z-transform, since samples in the frequency domain are located on the unit circle. Now we expand the sample location to arbitrary point in the z-plane. Hence, the N-point NDFT is defined as the frequency samples of the z-transform at arbitrary N points and expressed in the form

$$S_{ndft}(m) = \sum_{n=0}^{N-1} s_n\, z_m^{-n} \qquad (16)$$

where z_m is the complex points of interest that can be irregular spacing.

Eq. (16) can be rewritten as a matrix form $S = Ds$ where the matrix D and the vectors S and s are given as

$$S = \begin{bmatrix} S_{ndft}(z_0) \\ S_{ndft}(z_1) \\ \vdots \\ S_{ndft}(z_{N-1}) \end{bmatrix}, \quad s = \begin{bmatrix} s[0] \\ s[1] \\ \vdots \\ s[N-1] \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & z_0^{-1} & z_0^{-2} & \cdots & z_0^{-(N-1)} \\ 1 & z_1^{-1} & z_1^{-2} & \cdots & z_1^{-(N-1)} \\ 1 & \vdots & \vdots & \ddots & \vdots \\ 1 & z_{N-1}^{-1} & z_{N-1}^{-2} & \cdots & z_{N-1}^{-(N-1)} \end{bmatrix}$$

Computation of the matrix multiplication requires $O(N^2)$ operational complexity. A recursive algorithm called Horner's method can reduce memory size keeping only two multiplier coefficients. Eq. (16) can be written by

$$S_{ndft}(z_m) = z_m^{-(N-1)} \sum_{n=0}^{N-1} s_n\, z_m^{N-1-n} = z_m^{-(N-1)} A_m \qquad (17)$$

where

$$A_m = \sum_{n=0}^{N-1} s_n\, z_m^{N-1-n} = s_0 z_m^{N-1} + s_1 z_m^{N-2} + \cdots + s_{N-2} z_m + s_{N-1}$$

. It can also be expressed as a nested structure that can be calculated easily in a recursive algorithm :

$$A_m = \left(\cdots\left(s_0 z_m + s_1\right) z_m + \cdots\right) z_m + s_{N-1} \qquad (18)$$

The smallest nest output is $y_1 = y_0 z_m + s_1$ with the initial condition $y_0 = s_0$ and we can express Eq.(18) by the recursive difference equation as $y_n = y_{n-1} z_m + s_n$. When this algorithm has accumulated N times, we finally get the result $A_m = y_{N-1}$. Thus, only two multiplier coefficients (z_m and $z_m^{-(N-1)}$) are needed to calculate the m-th NDFT sample. reducing the memory size as well.

While the z-operator provides z-points in any part of z-plane, we can keep them on the unit-circle by introducing the exponential operator, i.e., the normal DFT. Computation in this case can be done by the Goertzel's algorithm using trigonometric series interpretation, requiring only three coefficients, $\cos w_m$, $\sin w_m$, and $e^{-j(N-1)w_m}$ for each NDFT sample. This method reduces the complexity down to $O(N)$. It is specialized for calculating the NDFT at points on the unit circle where w_m is the angular frequency. Now the z-operator in Eq. (17) is replaced with the exponents as

$$S_{ndft}(m) = e^{-j(N-1)w_m} A_m \qquad (19)$$

where $A_m = B_m + jC_m$ and $B_m = g_{N-1}\cos w_m + h_{N-1}$, $C_m = g_{\_N-1}\sin w_m$. The intermediate operators are defined as $g_i = 2\cos w_m g_{i-1} + h_{i-1}$ where $g_1 = s_0$ and $h_i = s_i - g_{i-1}$ where $h_1 = s_1$, $i = 2,3,...,N-1$.

3.1 Inverse NDFT

In general there is no simple inversion formula, hence one deals with the following reconstruction or recovery problem. Given the values $s_j \in \mathbb{C}$ $(j = 0,1,...,N-1)$ at nonequispaced points p_j, $j = 0,1,...,N-1$, the aim is to reconstruct a trigonometric polynomial $s(p_j)$ is close to the original sample s_j as

$$s(p_j) = \sum_k S_k e^{-j2\pi k p_j} \approx s_j, \;\; i.e., \;\; \boldsymbol{ES} \approx \boldsymbol{s} \qquad (20)$$

A standard method is to use the Moore-Penrose

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pseudoinverse solution which solves the general linear least squares problem

$$\|\boldsymbol{s}\|^2 \rightarrow \min \quad \text{subject to} \quad \|\boldsymbol{s}-\boldsymbol{ES}\|^2 = \min \qquad (21)$$

Of course, computing the pseudoinverse problem by the singular value decomposition is very expensive and no practical way at all.

As a more practical way, one can reduce the approximation error $r = \boldsymbol{s} - \boldsymbol{ES}$, by using the weighted approximation problem

$$\|\boldsymbol{s}-\boldsymbol{ES}\|\boldsymbol{W} = \left(\sum_{j=0}^{N-1} w_j \left|s_j - s(p_j)\right|^2\right)^{1/2} \rightarrow \min \qquad (22)$$

where $\boldsymbol{W}$ denotes diagonal weighting matrix.

Another method is to use the Lagrange interpolation technique to convert the given NDFT coefficients into the corresponding z-transform of the sequence. If this can be achieved, then the sequence can be identified as the coefficients of the z-transform. Using the Lagrange polynomial of order $N-1$, z-transform coefficients can be expressed as

$$S(z) = \sum_{m=0}^{N-1} \frac{L_m(z)}{L_m(z_m)} S_{ndft}[m] \qquad (23)$$

where $L_m(z) = \prod_{i \neq m} \left(1 - z_i z^{-1}\right), \quad m = 0,1,\ldots,N-1$.

4 Nonuniform sampling in the DFT domain

The aim of this work is to derive nonuniform sampling of signal in the frequency domain transformed by DFT, which will be transformed by the NDFT. For example, a signal is assumed to be mixed two sinusoidal sequences, i.e., 0.2π and 0.7π . DFT can be obtained by using the regular sample intervals. To compute the NDFT, we set nonuniform sample intervals based on the two main frequencies.

In this example, order of samples in the NDFT is nonlinearly allocated and concentrated on the main frequencies. This nonlinear resampling can be executed in the DFT domain as:

1. Execute DFT with equispaced samples.
2. Interpolate the sample intervals with certain ratio (more than 2) and reshape the amplitude spectrum based on the curve fitting, such as the cubic splined curvature.
3. Resample the interpolated samples to obtain the same number of total samples as in the equispaced case based on the centroid concept.

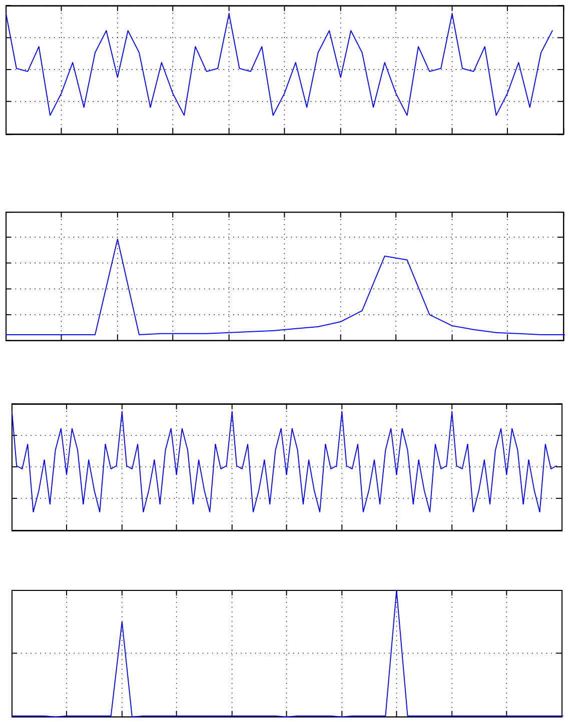

Fig. 1. (From top) Input sequence with 50 samples, DFT output, input sequence with 100 samples, and its DFT output

In Fig. 1, two different sequences are transformed by the conventional DFT, in which assumes the samples are infinite, i.e., the time duration is infinitely long, resulting in infinitely short response in the frequency domain. Otherwise, like in general applications, the DFT does not show proper frequency analysis, e.g., broad range of spectrum near 0.2π and 0.7π in Fig. 1b, while its shorter response with double number of samples in Fig. 1d.

As shown in Fig. 2, the DFT does not show the two frequencies well, i.e., two low amplitude spectrum is shown at around 0.7π . This is because the signal is band-limited in time. The NDFT results in optimal approximation of spectrum around the two frequencies by owing to dense sampling around the main spectra and rough sampling for other components.

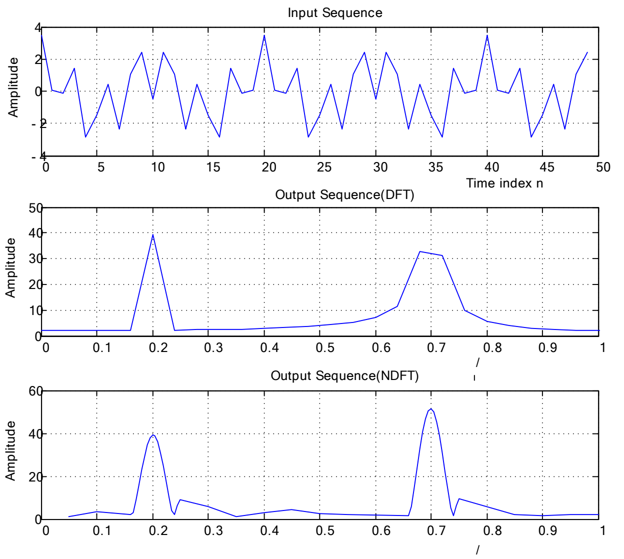

Fig. 2. (From top) Input sequence, DFT output, and NDFT output

5 Conclusion

We discussed any arbitrary sampled data could be transformed into frequency domain to get enhanced frequency analysis by eliminating the less important components and focusing only meaningful spectra. Nonuniform DFT is called for this purpose, that is derived from nonequispaced data. Nevertheless it is much far from perfect yet, remaining problems of fast computation such as the FFT, orthogonality which is a necessary condition for possible use in data recognition and compression, and particularly how to define nonequispaced sampling, it is worth of implementing the general transform with the inclusion of conventional unitary transform.

In this paper we proposed resampling method in the frequency domain to obtain nonequispaced transformation. DFT coefficients do not represent perfect spectra due to the band-limiting but can be utilized to approximate the main components. As for dense sampling at around the wanted spectrum it must be interpolated using spline method. Final resampling is achieved by rearranging the interpolated samples.

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