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About the Numerical Solution of a Stationary Transport Equation

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Abstract: - An algorithm for determining the solution of a boundary value problem for an integral-differential equation is presented. Using the method of decomposition for a transport equation in the stationary case and a plan-parallel geometry we obtain an approximate solution with an algorithm based on the variational form of the integral identity method. Several examples are included.

Key-Words: integral-differential equation, variational methods, integral identity method.

1 Introduction

In the neutrons transport literature, many authors paid attention to the numerical solutions obtained by the methods of Ritz and Galerkin, the method of least squares, [1],[5],[6],[8],[11], the method of finite elements and Nyström method [12].

In this paper we present an algorithm inspired by the variational form of the integral identity method, [5], applied to a diffusion equation. In the general case, this method is hard to use, but for any symmetry of the source function, it leads to an algorithm more flexible and computationally more efficient than the methods remind before. The numerical examples prove that the errors, which correspond to the approximate solutions, are minimum.

2 Problem formulation

In the stationary case, we consider a transport equation of the form

$$\mu \frac{\partial\varphi(x,\mu)}{\partial x} + \varphi(x,\mu) = \int_{-1}^{1} \varphi(x,\mu')d\mu' + f(x,\mu) \quad (1)$$

$$\forall(x,\mu) \in D_1 \times D_2 = [0,H] \times [-1,1],$$
$$D_2 = D_2' \cup D_2'' = [-1,0] \cup [0,1].$$

The boundary conditions are

$$\begin{aligned} \varphi(0,\mu) &= 0 \ \text{ if } \ \mu > 0 \\ \varphi(H,\mu) &= 0 \ \text{ if } \ \mu < 0 \end{aligned} \quad (2)$$

Here φ is the density of neutrons, which migrate in a direction defined by the angle α against Ox axis and we denote $\mu = \cos\alpha$. Let us consider the radioactive source f as an even function with respect to μ.
Using the notations:

$$\varphi^+ = \varphi(x,\mu) \text{ if } \mu > 0; \quad \varphi^- = \varphi(x,\mu) \text{ if } \mu < 0 \quad (3)$$

and substituting $\mu'' = -\mu'$, we get

$$\int_{-1}^{0} \varphi(x,\mu')d\mu' = \int_{0}^{1} \varphi(x,-\mu'')d\mu'' = \int_{0}^{1} \varphi^- d\mu''.$$

Then the conditions (2) become

$$\varphi^+(0,\mu) = 0, \ \varphi^-(H,\mu) = 0 \quad (4)$$

and the equation (1) can be written in the form

$$\mu \frac{\partial \varphi^+}{\partial x} + \varphi^+ = \int_0^1 (\varphi^+ + \varphi^-) d\mu' + f^+$$
$$- \mu \frac{\partial \varphi^-}{\partial x} + \varphi^- = \int_0^1 (\varphi^+ + \varphi^-) d\mu' + f^- \qquad (5)$$

Adding and subtracting the equations (5) and introducing the notations:

$$u = \frac{1}{2}\left(\varphi^+ + \varphi^-\right), \quad v = \frac{1}{2}\left(\varphi^+ - \varphi^-\right)$$
$$g = \frac{1}{2}\left(f^+ + f^-\right), \quad r = \frac{1}{2}\left(f^+ - f^-\right) = 0 \qquad (6)$$

we obtain the following system

$$\mu \frac{\partial v}{\partial x} + u = 2\int_0^1 u\, d\mu + g \quad (a)$$
$$\mu \frac{\partial u}{\partial x} + v = 0 \quad (b) \qquad (7)$$

The boundary conditions become

$$u + v = 0 \quad \text{for} \quad x = 0,$$
$$u - v = 0 \quad \text{for} \quad x = H. \qquad (8)$$

Now, we find v from the second equation of (7) and using the first equation, we rewrite the problem (7)-(8) in the following form

$$- \mu^2 \frac{\partial^2 u}{\partial x^2} + u = 2\int_0^1 u d\mu + g \qquad (9)$$

$$\left(u - \mu \frac{\partial u}{\partial x}\right)\Bigg|_{x=0} = \left(u + \mu \frac{\partial u}{\partial x}\right)\Bigg|_{x=H} = 0 \qquad (10)$$

In order to get a solution of the problem (9)-(10), we consider two points systems on the x axis:

- a principal system: $\{x_k\} = \Delta_1'$, $k \in \{0, 1, \ldots, N\}$, with $x_0 = 0$, $x_N = H$ and $h = x_{k+1} - x_k$;
- a secondary system, $\{x_{k+1/2}\} = \Delta_1''$, $k \in \{0,1,2,\ldots, N-1\}$, which verifies the inequalities: $x_{k-1/2} < x_k < x_{k+1/2}$, where

$$x_{k+1/2} = (x_k + x_{k+1})/2$$

and

$$0 = x_0 < x_{1/2} < \ldots < x_{N-1/2} < x_N = H.$$

Besides this, let $\Delta_2 = \{\mu_l\}$, $l \in \{0,1,\ldots, L\}$ be a partition of the interval $D_2'' = [0,1]$ and $\tau = \mu_{l+1} - \mu_l$, $l \in \{0,1,\ldots, L-1\}$.

Further on, we consider H =1. For every value $\mu_l \in \Delta_2$, the problem (9)-(10) becomes:

$$- \mu^2 \frac{d^2 u(x,\mu_l)}{dx^2} + u(x,\mu_l) = f_1(x,\mu_l) \qquad (11)$$

where

$$f_1(x,\mu_l) = S(x) + g(x,\mu_l), \; S(x) = 2\int_0^1 u(x,\mu) d\mu$$

and

$$\left(u(x,\mu_l) - \mu_l \frac{du(x,\mu_l)}{dx}\right)\Bigg|_{x=0} = 0$$
$$\left(u(x,\mu_l) + \mu_l \frac{du(x,\mu_l)}{dx}\right)\Bigg|_{x=1} = 0 \qquad (12)$$

Here we assume $u \in L_2$ ([0, 1]), the Hilbert space with the scalar product defined by formula

$$(u,v) = \int_0^1 w(x) v(x) dx \qquad (13)$$

Now (11)-(12) is a boundary problem for a one-dimensional diffusion equation (11). Integrating (11) with respect to x on the intervals: $(x_{k-1/2}, x_{k+1/2})$, we obtain

$$- J_{k+1/2} + J_{k-1/2} + \int_{x_{k-1/2}}^{x_{k+1/2}} (u - f_1) dx = 0 \qquad (14)$$

where

$$J_{k\pm1/2} = J(x_{k\pm1/2}), \quad J(x,\mu_l) = \mu_l^2 \frac{du(x,\mu_l)}{dx}.$$

We find $J_{k-1/2}$ integrating (11) on the interval $(x_{k-1/2}, x)$. We get

$$\mu^2 \frac{du(x,\mu_l)}{dx} = J_{k-1/2} + \int_{x_{k-1/2}}^{x} (u - f_1) dx \qquad (15)$$

Then, dividing (15) by μ^2 and integrating on (x_{k-1}, x_k) we have

$$u_k - u_{k-1} = J_{k-1/2} \int_{x_{k-1}}^{x_k} \frac{dx}{\mu_l^2} + \int_{x_{k-1}}^{x_k} \frac{dx}{\mu_l^2} \int_{x_{k-1/2}}^{x} (u - f_1) d\xi \quad (16)$$

Finally, we get

$$J_{k-1/2} = \frac{\mu_l^2}{h} \left[u_k - u_{k-1} - \frac{1}{\mu_l^2} \int_{x_{k-1}}^{x_k} dx \int_{x_{k-1/2}}^{x} (u - f_1) d\xi \right] \quad (17)$$

In a similar manner, we obtain $J_{k+1/2}$ replacing k by k+1. Consequently, the equality (14) becomes:

$$\mu^2 \left(\frac{u_k - u_{k+1}}{h} + \frac{u_k - u_{k-1}}{h} \right) + \int_{x_{k-1/2}}^{x_{k+1/2}} (u - f_1) dx =$$
$$= -\frac{1}{h} \int_{x_k}^{x_{k+1}} dx \int_{x_{k+1/2}}^{x} (u - f_1) dx + \frac{1}{h} \int_{x_{k-1}}^{x_k} dx \int_{x_{k-1/2}}^{x} (u - f_1) dx \quad (18)$$

Now we shall denote

$$\psi(x) = u(x, \mu_l) - f_1(x, \mu_l)$$
$$\rho_k(x) = (x - x_k)/h . \quad (19)$$

Appling the method of integration by parts we obtain

$$-\frac{1}{h} \int_{x_k}^{x_{k+1}} d\left(\int_{x_k}^{x} d\alpha \right) \int_{x_{k+1/2}}^{x} \psi(\xi) d\xi =$$
$$= -\frac{1}{h} (x - x_k) \int_{x_{k+1/2}}^{x} \psi(\xi) d\xi \Bigg|_{x_k}^{x_{k+1}} + \frac{1}{h} \int_{x_k}^{x_{k+1}} \left(\int_{x_k}^{x} d\alpha \right) \psi(x) dx =$$
$$= -\frac{1}{h} \left[h \int_{x_{k+1/2}}^{x_{k+1}} \psi(\xi) d\xi - \int_{x_k}^{x_{k+1}} (x - x_k) \psi(x) dx \right] =$$
$$= -\int_{x_{k+1/2}}^{x_{k+1}} \psi(\xi) d\xi + \int_{x_k}^{x_{k+1}} \rho_k(x) \psi(x) dx \quad (20)$$

Analogously, if we denote

$$\tilde{\rho}_k(x) = \frac{x_k - x}{h} \quad (21)$$

the equation (18) is now of the form

$$\mu_l^2 \left(\frac{u_k - u_{k+1}}{h} + \frac{u_k - u_{k-1}}{h} \right) + \int_{x_{k-1}}^{x_k} (1 - \tilde{\rho}_k(x)) \psi(x) dx +$$
$$+ \int_{x_k}^{x_{k+1}} (1 - \rho_k(x)) \psi(x) dx = 0 \quad (22)$$

It should be observed that the integral from the left-hand side of (18) was decomposed in the intervals: $(x_{k-1/2}, x_k) \cup (x_k, x_{k+1/2})$.

Let us introduce the functions:

$$Q_k(x) = \begin{cases} \frac{1 - \tilde{\rho}_k}{\sqrt{h}} = \frac{x - x_{k-1}}{h\sqrt{h}}, & x \in [x_{k-1}, x_k] \\ \frac{1 - \rho_k}{\sqrt{h}} = \frac{x_{k+1} - x}{h\sqrt{h}}, & x \in [x_k, x_{k+1}] \\ 0, & x \notin [x_{k-1}, x_{k+1}] \end{cases} \quad (23)$$

where

$$Q_k(x_k) = \frac{1}{\sqrt{h}} .$$

Then, using the scalar product, the equations (18) can be rewritten in the form

$$\mu_l^2 \left(\frac{u(x_k, \mu_l) - u(x_{k+1}, \mu_l)}{h} + \frac{u(x_k, \mu_l) - u(x_{k-1}, \mu_l)}{h} \right) +$$
$$+ (u, \sqrt{h} Q_k) = (f_1, \sqrt{h} Q_k), \ k \in \{1, ..., N-1\} \quad (24)$$

On the other hand, we observe that

$$\left(\mu_l^2 \frac{du(x, \mu_l)}{dx}, \frac{dQ_k}{dx} \right) = \mu_l^2 \int_{x_{k-1}}^{x_k} \frac{du(x, \mu_l)}{dx} \cdot \frac{1}{h\sqrt{h}} dx -$$
$$- \mu_l^2 \int_{x_k}^{x_{k+1}} \frac{du(x, \mu_l)}{dx} \cdot \frac{1}{h\sqrt{h}} dx = \quad (25)$$
$$= \mu_l^2 \left(\frac{u_k - u_{k-1}}{h\sqrt{h}} - \frac{u_{k+1} - u_k}{h\sqrt{h}} \right), k \in \{1, 2, ..., N-1\}$$

and (24) can be written in the following form:

$$\left(\mu_l^2 \frac{du}{dx}, \frac{dQ_k}{dx} \right) + (u, Q_k) = (f_1, Q_k), \ k \in \{1, 2, .., N-1\} \quad (26)$$

Hence, the identity (18) is replaced by the relations (26). This allows us to consider the integral identity method as a variational method and the equations (26) may be used for determining the approximate solutions using a sequence of coordinate functions. It should be noted that the equations (26) coincide with the relations obtained by Galerkin method, where $Q_k(x)$ are the coordinate functions. Then, the solution of the system (24) can be defined in the following way

$$\tilde{u}(x,\mu_l) = \sum_{k=1}^{N-1} a_k(\mu_l) Q_k(x) \qquad (27)$$

where: $a_k(\mu_l) = 2\beta_k \mu_l^2$.

We shall now determine the coefficients β_k from the condition that (27) be a solution of the system (24). Also, the boundary conditions must be satisfied.

Since the functions u are linear with respect to x, we get from (24) for k segment (x_{x-1}, x_{k+1}) and $x_k = kh$:

$$\mu_l^2 \left(\frac{\tilde{u}_k - \tilde{u}_{k+1}}{h\sqrt{h}} + \frac{\tilde{u}_k - \tilde{u}_{k-1}}{h\sqrt{h}} \right) = \frac{\mu^2}{h^2}\left(2\tilde{u}_k - \tilde{u}_{k-1} - \tilde{u}_{k+1}\right) = -\frac{2\mu^4}{h^2}\left(\beta_{k-1} - 2\beta_k + \beta_{k+1}\right) \qquad (28)$$

$$\left(\tilde{u}, Q_k\right) = \int_{x_{k-1}}^{x_k} \left(\sum_{j=1}^{N} 2\beta_j \mu_l^2 Q_j(x) \right) Q_k(x)dx + \int_{x_k}^{x_{k+1}} \left(\sum_{j=1}^{N} 2\beta_j \mu_l^2 Q_j(x) \right) Q_k(x)dx =$$

$$= 2\mu_l^2 \int_{x_{k-1}}^{x_k} \left[\beta_{k-1} \frac{x_k - x}{h\sqrt{h}} \cdot \frac{x - x_{k-1}}{h\sqrt{h}} + \beta_k \left(\frac{x - x_{k-1}}{h\sqrt{h}} \right)^2 \right] dx + 2\mu_l^2 \int_{x_k}^{x_{k+1}} \left[\beta_{k+1} \frac{x - x_k}{h\sqrt{h}} \cdot \frac{x_{k+1} - x}{h\sqrt{h}} + \beta_k \left(\frac{x_{k+1} - x}{h\sqrt{h}} \right)^2 \right] dx =$$

$$= \frac{\mu_l^3}{3}\left(\beta_{k-1} + 4\beta_k + \beta_{k+1}\right) \qquad (29)$$

$$S(x) = 2\int_0^1 \tilde{u}(x,\mu)d\mu = 2\sum_{j=1}^{N}\int_0^1 2\beta_j \mu^2 Q_j(x)d\mu = \frac{4}{3}\sum_{j=1}^{N} \beta_j Q_j(x)$$

and

$$\left(S(x), Q_k(x)\right) = \frac{4}{3}\int_{x_{k-1}}^{x_k} \left(\sum_{j=1}^{N} \beta_j Q_j \right) Q_k(x)dx + \frac{4}{3}\int_{x_k}^{x_{k+1}} \left(\sum_{j=1}^{N} \beta_j Q_j \right) Q_k(x)dx = \frac{2}{9}\left(\beta_{k-1} + 4\beta_k + \beta_{k+1}\right) \qquad (30)$$

Thus, we arrive at the following system:

$$-\frac{2\mu_l^4}{h^2}\left(\beta_{k-1} - 2\beta_k + \beta_{k+1}\right) + \frac{3\mu_l^2 - 2}{9}\left(\beta_{k-1} + 4\beta_k + \beta_{k+1}\right) = \gamma_k \qquad (31)$$

$$k \in \{1,\ldots, N-1\}$$

where

$$\gamma_k = \left(g, Q_k\right) = \int_{x_{k-1}}^{x_{k+1}} g(x)Q_k(x)dx..$$

Now we denote:

$$a_{k-1,k} = -\frac{2\mu_l^4}{h^2} + \frac{3\mu_l^2 - 2}{9}, \; k \in \{2,\ldots, N-1\}$$

$$a_{k,k} = \frac{4\mu_l^4}{h^2} + \frac{4(3\mu_l^2 - 2)}{9}, \; k \in \{1,\ldots, N-1\} \qquad (32)$$

$$a_{k,k+1} = a_{k-1,k}, \; k \in \{1,\ldots, N-2\}$$

The formula (31) can be written as a matrix equation

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{\Gamma} \qquad (33)$$

where

- matrix **A** is of the form

$$\begin{bmatrix} a_{1,1} & a_{1,2} & 0 & \cdots & 0 & 0 & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{N-2,N-3} & a_{N-2,N-2} & a_{N-2,N-1} \\ 0 & 0 & 0 & \cdots & 0 & a_{N-1,N-2} & a_{N-1,N-1} \end{bmatrix}$$

- **B** is a column matrix:

$$\left[\beta_1 \;\; \beta_2 \;\; \cdots \;\; \beta_{N-1}\right]^T$$

- **Γ** is a column matrix:

$$\left[\gamma_1 \;\; \gamma_2 \;\; \cdots \;\; \gamma_{N-1}\right]^T.$$

Here one property of approximation (27) should be noted. The functions Q_{k-1} and Q_{k+1} are orthogonal, since at such places where one of these is nonzero, the other is equal to zero. Thus, the basis introduced is “almost orthogonal”. This is the reason behind the appearance of band in the matrix **A**.

Solving the system of equations (33) we can find the values of coefficients β_k, i.e. we can construct by (27) the solution u of (11)-(12).
Let us now consider

$$x_k = y_0 < y_1 < \ldots < y_m = x_{k+1}$$

for every interval (x_k, x_{k+1}), $k = 0, 1, \ldots, N-1$. Using the $Q_k(x)$, we can find the values of $\tilde{u}_{k,j}$, $k \in \{1,2,\ldots,N-1\}$, $j \in \{1,2,\ldots,m\}$. In order to get v, we use $\tilde{u}_{k,j}$ and the numerical derivative for equation (7*b*):

$$\tilde{v}_{k,j} = -\frac{\tilde{u}_{k,j+1} - \tilde{u}_{k,j-1}}{2(h/m)} \cdot \mu_l^2, \quad k \in \{1,2,\ldots,N\},\ j \in \{1,2,\ldots,m-1\} \qquad (34)$$

$$\tilde{v}_{k,0} = -\frac{\tilde{u}_{k+1,1} - \tilde{u}_{k,m-1}}{2(h/m)} \cdot \mu_l^2,\ k \in \{1,2,\ldots,N-1\}$$

$$v_{1,0} = -\frac{\tilde{u}_{1,1} \cdot \mu_l^2}{h/m}, \qquad v_{N,m} = -\frac{\tilde{u}_{N,m} - \tilde{u}_{N,m-1}}{h/m}$$

where

$$\tilde{u}_{k,j} = 2\mu_l^2\left(\beta_{k-1}\frac{x_{k+1} - y_j}{h\sqrt{h}} + \beta_k \frac{y_j - x_{k-1}}{h\sqrt{h}}\right), \quad k \in \{1,2,\ldots,N-1\},\ j \in \{0,1,\ldots,m\} \qquad (35)$$

According to the continuity of function u we get

$$\tilde{u}_{k-1,m} = \tilde{u}_{k,0},\ k \in \{1,2,\ldots,N\}. \qquad (36)$$

Finally, the values of φ obtained by this algorithm will be

$$\tilde{\varphi}^+_{k,j} = u_{k,j} + v_{k,j} \ \text{ for } \ \mu_l > 0$$
$$\tilde{\varphi}^-_{k,j} = u_{k,j} - v_{k,j} \ \text{ for } \ \mu_l < 0 \qquad (37)$$
$$k \in \{1,2,\ldots,N\},\ j \in \{0,1,\ldots,m\}.$$

3 Numerical examples

Let us consider the stationary transport equation

$$\mu\frac{\partial\varphi(x,\mu)}{\partial x} + \varphi(x,\mu) = \int_{-1}^{1}\varphi(x,\mu')d\mu' + f(x,\mu) \qquad (38)$$
$$\forall (x,\mu) \in D_1 \times D_2,\ D_1 = [0,1], D_2 = [-1,1]$$

with the boundary conditions

$$u(0,\mu) = 0,\ \mu > 0; \quad u(1,\mu) = 0,\ \mu < 0. \qquad (39)$$

Now, we choose an even function f with respect μ of the form

$$f(x,\mu) = -2\pi^2\mu^4\cos(2\pi x) + \mu^2\sin^2(\pi x) \qquad (40)$$

Hence, $r = (f^+ - f^-)/2 = 0$ and we obtain $g = f$.
We break up the closed interval D_1 into $N = 8$ segments of length $h = 1/8$ and for $D_2'' = [0,1]$ we have $L = 4$. Some computational results will illustrate the application of above algorithm.
Let us now consider that $\mu_l = 1/2$. Using (39) we get

$$\gamma_k = \int_{x_{k-1}}^{x_{k+1}} g(x)Q_k(x)dx = $$
$$= A_1\frac{\cos(2k\pi h)\sin^2(\pi h)}{\pi^2} + B_1 h^2,\ k \in \{1,..,N-1\}$$

where

$$A_1 = \frac{1}{h\sqrt{h}}\left(\frac{2-3\mu^2}{6} - 2\mu^4\pi^2\right), B_1 = \frac{3\mu^2 - 2}{6h\sqrt{h}},\ k \in \{1,2,\ldots,7\}. \qquad (41)$$
$$\gamma_0 = \gamma_8 = A_1\frac{\sin^2(\pi h)}{2\pi^2} + B_1\frac{h^2}{2}.$$

In this case, the matrices of equation (33) are of the form

$$\mathbf{A} = \begin{bmatrix} 15.4 & -8.14 & 0 & 0 & 0 & 0 & 0 \\ -8.14 & 15.4 & -8.14 & 0 & 0 & 0 & 0 \\ 0 & -8.14 & 15.4 & -8.14 & 0 & 0 & 0 \\ 0 & 0 & -8.14 & 15.4 & -8.14 & 0 & 0 \\ 0 & 0 & 0 & -8.14 & 15.4 & -8.14 & 0 \\ 0 & 0 & 0 & 0 & -8.14 & 15.4 & -8.14 \\ 0 & 0 & 0 & 0 & 0 & -8.14 & 15.4 \end{bmatrix},$$

$$\Gamma = [-0.21\ -0.074\ 0.17\ 0.27\ 0.17\ -0.073\ -021]^T$$

Solving (33), we obtain **B**, the matrix of coefficients β_k

$$\mathbf{B} = [0.052\ 0.123\ 0.2\ 0.22\ 0.2\ 0.123\ 0.052]^T.$$

In the following, each interval, $[x_k, x_{k+1}]$ is divided into subintervals $[y_j, y_{j+1}]$, where $x_k = y_0 < y_1 < y_2 < y_3 < y_4 = x_{k+1}$, $k \in \{1,2,\ldots,7\}$, $y_{j+1} - y_j = h/4$, $j \in \{0,1,2,3\}$, $m = 4$ and

$u_{k,0} = u_{k-1,4}$, $k \in \{2,...,8\}$. Table 1 shows the values $u_{k,j}$, which were calculated with the help of the formula (35) :

Table 1

k \ j	**0**	**1**	**2**	**3**	**4**
0	0	0	0	0	0
1	0	0.018	0.036	0.055	0.073
2	0.073	0.098	0.124	0.149	0.175
3	0.175	0.2	0.223	0.247	0.271
4	0.271	0.281	0.291	0.3	0.311
5	0.311	0.3	0.291	0.281	0.272
6	0.272	0.247	0.223	0.2	0.175
7	0.175	0.149	0.124	0.099	0.073
8	0.073	0.055	0.037	0.018	0.

Table 2

k \ j	**0**	**1**	**2**	**3**	**4**
0	0	0	0	0	0
1	0	-0.29	-0.29	-0.29	-0.35
2	-0.35	-0.41	-0.41	-0.41	-0.4
3	-0.4	-0.39	-0.39	-0.39	-0.27
4	-0.27	-0.16	-0.16	-0.16	0
5	0	0.16	0.16	0.16	0.27
6	0.27	0.39	0.39	0.39	0.4
7	0.4	0.41	0.41	0.41	0.35
8	0.35	0.29	0.29	0.29	0.

Table 2 shown the values of v, calculated with the help of the relations (34).

Approximate values of the solution of equation (11), where g is definite by (38), were compared with these obtained by exact solution:

$$ue(x, \mu_l) = \mu_l^2 \sin^2(\pi x) . \qquad (42)$$

Finally, the solution $\varphi_{k,j}$ of the boundary problem (38) - (40) for μ_l = 1/2 was computed from (37). Hence

$$\varphi_{k,j} = \tilde{u}_{k,j} + \tilde{v}_{k,j}$$

$k \in \{1,2,...,7\}$, $j \in \{0,1,2,3,4\}$.

Table 3 shows the values of $\tilde{u}(x_k, \mu_l), \tilde{v}(x_k, \mu_l)$ φ+(x_k, μ_l), $k \in \{1,2,..,8\}$ for different values of μ..

The numerical solution u of the boundary problem (11)-(12) has been compared with the exact solution ue and an estimation of the correspond approximations:

$$\varepsilon = u - ue$$

have been given.

μ = 1/4 *Table* 3

k	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
u	-0.0026	0.007	0.058	0.09	0.058	0.007	-0.0024	0
v	-0.007	-0.06	-0.083	0	0.083	0.06	0.007	0
φ	-0.01	-0.054	-0.024	0.09	0.14	0.068	0.005	0
ue	0.009	0.031	0.053	0.062	0.053	0.031	0.009	0
ε	-0.012	-0.024	0.005	0.027	0.005	-0.024	-0.012	0

μ = 1/2 *Table* 4

k	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
u	0.073	0.175	0.271	0.311	0.272	0.175	0.073	0
v	-0.35	-0.4	-0.27	0.	0.27	0.4	0.35	0
φ	-0.276	-0.222	-0.001	0.31	0.543	0.57	0.423	0
ue	0.037	0.125	0.213	0.25	0.214	0.125	0.037	0
ε	0.036	0.05	0.058	0.061	0.058	0.05	0.036	0

μ = 3/4 *Table* 5

k	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
u	0.117	0.317	0.515	0.6	0.516	0.317	0.118	0
v	0.95	-1.19	-0.845	-0.002	0.843	1.195	0.95	0
φ	-0.83	-0.878	-0.33	0.6	1.36	1.512	1.07	0

ue	0.082	0.28	0.48	0.56	0.48	0.28	0.083	0
ε	0.035	0.035	0.036	0.036	0.035	0.035	0.035	0

$\mu = 1$ *Table* 6

k	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
u	0.205	0.56	0.91	1.06	0.912	0.56	0.206	0
v	-2.23	-2.82	-2	-0.004	2	2.82	2.24	0
φ	-2.026	-2.265	1.088	1.054	2.906	3.38	2.44	0
ue	0.146	0.5	0.853	1	0.854	0.5	0.147	0
ε	0.059	0.058	0.058	0.058	0.058	0.058	0.058	0

4 Conclusions

Analyzing the results of numerical examples, we find that the difference

$$\left| ue - \tilde{u} \right| \leq 0.06 \cong 2h\sqrt{\mu h}$$

Let it be considered an interpolation polynomial of solution $u(x, \mu_l)$:

$$U(x, \mu_l) = \sum_{k=1}^{N-1} u(x_k, \mu_l) Q_k(x) \sqrt{h} \qquad (43)$$

hence

$$U(x_k,\mu_l) = u(x_k,\mu_l),\ k \in \{0,1,\ldots,N\}. \qquad (44)$$

Further on, we noted for a fixed value μ_l:
$U(x) = U(x,\mu_l)$ and $u(x) = u(x,\mu_l)$.
To obtain an estimate of the error occurring as a result of replacing the function $u(x)$ by $U(x)$, we have computed for every $x \in [x_{k-1}, x_k]$, the difference:

$$\begin{aligned} u(x) - U(x) &= \int_{x_{k-1}}^{x} \frac{d}{dt}(u(t) - U(t))(t')dt' = \\ &= \int_{x_{k-1}}^{x} \frac{du}{dt}(t')dt' - \int_{x_{k-1}}^{x} \frac{dt'}{h} \int_{x_{k-1}}^{x_k} \frac{du}{dt}(t'')dt'' = \\ &\leq \frac{1}{h} \int_{x_{k-1}}^{x_k} dt'' \int_{x_{k-1}}^{x} \left[\frac{du}{dt}(t') - \frac{du}{dt}(t'') \right] dt' = \\ &= \frac{1}{h} \int_{x_{k-1}}^{x_k} dt'' \int_{x_{k-1}}^{x} dt' \int_{t''}^{t'} \frac{d^2u}{d\xi^2}(\xi) d\xi \end{aligned} \qquad (45)$$

This leads to the inequality

$$\left| u(x) - U(x) \right| \leq \frac{h^2}{h} \int_{x_{k-1}}^{x_k} \left| \frac{d^2u}{d\xi^2} \right| d\xi,\ \forall x \in \left[x_{k-1}, x_k \right]$$

hence

$$\max_{[x_{k-1}, x_k]} \left| u(x) - U(x) \right| \leq h \max_{[x_{k-1}, x_k]} \left| \frac{d^2u}{d\xi^2} \right| h =$$

$$= h^2 \left\| \frac{d^2u}{d\xi^2} \right\|_{L_\infty[x_{k-1}, x_k]} \leq M h^2 \left\| f_1 \right\|_{L_\infty[x_{k-1}, x_k]}.$$

If $\dfrac{d^2u}{dx^2} \in L_{\infty[0,1]}$, when $H = 1$, we also get

$$\left\| u - U \right\|_{C[0,1]} \leq Mh^2 \left\| \frac{d^2u}{dx^2} \right\|_{L_\infty[0,1]} \leq$$

$$\leq Mh^2 \left\| f_1 \right\|_{L_\infty[0,1]}.$$

Thus, we have obtained that the approximation of u is of the order h^2.

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