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Modified Theory of Laminar Boundary Layer Flow Over a Flat Plate

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Abstract:- Scale-invariant forms of the mass, energy, and linear momentum conservation equations in reactive fields are described. The modified form of the equation of motion is then solved for the classical *Blasius* problem of laminar flow over a flat plate. The predicted velocity profile is found to be in excellent agreement with the more recent experimental data of *Dhawan*, and in close agreement with the earlier experimental data of *Nikuradse* as well as the numerical calculation of *Blasius.*

Key-Words: - *Theory of laminar flow over a flat plate. Laminar boundary layer theory.*

1 Introduction

The universality of turbulent phenomena from stochastic quantum fields to classical hydrodynamic fields resulted in recent introduction of a scale-invariant model of statistical mechanics and its application to the field of thermodynamics [4]. The implications of the model to the study of transport phenomena and invariant forms of conservation equations have also been addressed [5, 6]. In the present study, following *Blasius* [10], the modified form of the equation of motion is solved for the problem of laminar flow over a flat plate. The predicted velocity profile is shown to be in close agreement with the early experimental data of *Nikuradse* [11] and in excellent agreement with the more recent experimental data of *Dhawan* [12].

2 Scale-Invariant Forms of the Conservation Equations for Reactive Fields

Following the classical methods [1-3], the invariant definitions of the density ρ_β, and the velocity of *atom* $\mathbf{u}_\beta$, *element* $\mathbf{v}_\beta$, and *system* $\mathbf{w}_\beta$ at the scale β are given as [4]

$$\rho_\beta = n_\beta m_\beta = m_\beta \int f_\beta du_\beta \quad , \quad \mathbf{u}_\beta = \mathbf{v}_{\beta-1} \qquad (1)$$

$$\mathbf{v}_\beta = \rho_\beta^{-1} m_\beta \int \mathbf{u}_\beta f_\beta d\mathbf{u}_\beta \quad , \quad \mathbf{w}_\beta = \mathbf{v}_{\beta+1} \qquad (2)$$

The scale-invariant model of statistical mechanics for equilibrium fields of . . . eddy-, cluster-, molecular-, atomic-dynamics . . . at the scale β = e, c, m, a, and the corresponding *non-equilibrium* laminar flow fields are schematically shown in Fig.1. Each statistical field, described by a distribution function $f_\beta(\mathbf{u}_\beta) = f_\beta(\mathbf{r}_\beta, \mathbf{u}_\beta, t_\beta)\, d\mathbf{r}_\beta d\mathbf{u}_\beta$, defines a "system" that is composed of an ensemble of "elements", each element is composed of an ensemble of small particles viewed as *point-mass* "atoms". The element (system) of the smaller scale (β) becomes the atom (element) of the larger scale (β+1). The three characteristic length scales associated with the free paths of atoms, and elements, and the size of the system at any scale β are ($l_\beta = \lambda_{\beta-1}$, λ_β, $L_\beta = \lambda_{\beta+1}$) where $\lambda_\beta = \langle l^2_\beta \rangle^{1/2}$ is the cluster length that is also equal to the mean-free-path of the atoms [5].

The invariant definitions of the peculiar and the diffusion velocities have been introduced as [4]

$$\mathbf{V}'_\beta = \mathbf{u}_\beta - \mathbf{v}_\beta \quad , \quad \mathbf{V}_\beta = \mathbf{v}_\beta - \mathbf{w}_\beta = \mathbf{V}'_{\beta+1} \qquad (3)$$

It is noted that according to (3) the diffusion velocity at the scale β becomes the peculiar velocity at the next larger scale β+1. For the equilibrium statistical fields shown on the left side of Fig.1, $f_\beta(\mathbf{u}_\beta)$ will be the *Maxwell-Boltzmann* distribution function. The “atomic”, the local “element”, and the convective “system” velocities ($\mathbf{u}_\beta$, $\mathbf{v}_\beta$, $\mathbf{w}_\beta$) defined in (1)-(2) at each scale within the hierarchy are also shown on the right side of Fig.1.

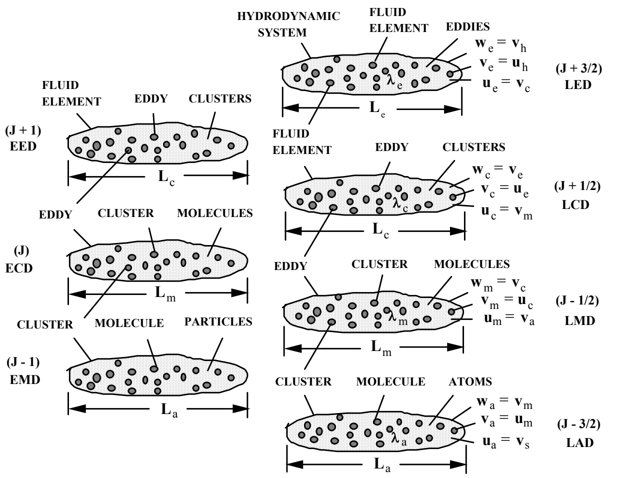

Fig.1 Hierarchy of statistical fields for equilibrium eddy-, cluster-, and molecular-dynamic scales and the associated laminar flow fields.

Next, following the classical methods [1-3], the scale-invariant forms of mass, thermal energy, and linear momentum conservation equations at scale β are given as [5, 6]

$$\frac{\partial \rho_\beta}{\partial t} + \nabla.\left(\rho_\beta \mathbf{v}_\beta\right) = \Omega_\beta \tag{4}$$

$$\frac{\partial \varepsilon_\beta}{\partial t} + \nabla.\left(\varepsilon_\beta \mathbf{v}_\beta\right) = 0 \tag{5}$$

$$\frac{\partial \mathbf{p}_\beta}{\partial t} + \nabla.\left(\mathbf{p}_\beta \mathbf{v}_\beta\right) = 0 \tag{6}$$

that involve the *volumetric density* of thermal energy $\varepsilon_\beta = \rho_\beta h_\beta$ and linear momentum $\mathbf{p}_\beta = \rho_\beta \mathbf{v}_\beta$. Also, Ω_β is the chemical reaction rate and h_β is the absolute enthalpy.

The local velocity $\mathbf{v}_\beta$ in (4)-(6) is expressed in terms of the convective $\mathbf{w}_\beta = \langle \mathbf{v}_\beta \rangle$ and the diffusive $\mathbf{V}_\beta$ velocities [5]

$$\mathbf{v}_\beta = \mathbf{w}_\beta + \mathbf{V}_{\beta g} \quad , \quad \mathbf{V}_{\beta g} = -D_\beta \nabla \ln(\rho_\beta) \tag{7a}$$

$$\mathbf{v}_\beta = \mathbf{w}_\beta + \mathbf{V}_{\beta tg} \quad , \quad \mathbf{V}_{\beta tg} = -\alpha_\beta \nabla \ln(\varepsilon_\beta) \tag{7b}$$

$$\mathbf{v}_\beta = \mathbf{w}_\beta + \mathbf{V}_{\beta hg} \quad , \quad \mathbf{V}_{\beta hg} = -\nu_\beta \nabla \ln(\mathbf{p}_\beta) \tag{7c}$$

where ($\mathbf{V}_{\beta g}$, $\mathbf{V}_{\beta tg}$, $\mathbf{V}_{\beta hg}$) are respectively the diffusive, the thermo-diffusive, the linear hydro-diffusive velocities. For unity *Schmidt* and *Prandtl* numbers, one may express

$$\mathbf{V}_{\beta tg} = \mathbf{V}_{\beta g} + \mathbf{V}_{\beta t} \quad , \quad \mathbf{V}_{\beta t} = -\alpha_\beta \nabla \ln(h_\beta) \tag{8a}$$

$$\mathbf{V}_{\beta hg} = \mathbf{V}_{\beta g} + \mathbf{V}_{\beta h} \quad , \quad \mathbf{V}_{\beta h} = -\nu_\beta \nabla \ln(\mathbf{v}_\beta) \tag{8b}$$

that involve the thermal $\mathbf{V}_{\beta t}$, and linear hydrodynamic $\mathbf{V}_{\beta h}$ diffusion velocities [5]. Since for an ideal gas $h_\beta = c_{p\beta} T_\beta$, when $c_{p\beta}$ is constant and $T = T_\beta$, Eq.(8a) reduces to the *Fourier* law of heat conduction

$$\mathbf{q}_\beta = \rho_\beta h_\beta \mathbf{V}_{\beta t} = -\kappa_\beta \nabla T \tag{9}$$

where κ_β and $\alpha_\beta = \kappa_\beta / (\rho_\beta c_{p\beta})$ are the thermal conductivity and diffusivity. Similarly, (8b) may be identified as the shear stress associated with diffusional flux of linear momentum and expressed by the generalized *Newton* law of viscosity [5]

$$\boldsymbol{\tau}_{ij\beta} = \rho_\beta \mathbf{v}_{j\beta} \mathbf{V}_{ij\beta h} = -\mu_\beta \partial \mathbf{v}_{j\beta} / \partial \mathbf{x}_i \tag{10}$$

Substitutions from (7a)-(7c) into (4)-(6), neglecting cross-diffusion terms and assuming constant transport coefficients with $Sc_\beta = Pr_\beta = 1$, result in [5, 6]

$$\frac{\partial \rho_\beta}{\partial t} + \mathbf{w}_\beta . \nabla \rho_\beta - D_\beta \nabla^2 \rho_\beta = \Omega_\beta \tag{11}$$

$$h_\beta \left[\frac{\partial \rho_\beta}{\partial t} + \mathbf{w}_\beta . \nabla \rho_\beta - D_\beta \nabla^2 \rho_\beta \right] + \rho_\beta \left[\frac{\partial h_\beta}{\partial t} + \mathbf{w}_\beta . \nabla h_\beta - \alpha_\beta \nabla^2 h_\beta \right] = 0 \tag{12}$$

$$\mathbf{v}_\beta \left[\frac{\partial \rho_\beta}{\partial t} + \mathbf{w}_\beta . \nabla \rho_\beta - D_\beta \nabla^2 \rho_\beta \right] + \rho_\beta \left[\frac{\partial \mathbf{v}_\beta}{\partial t} + \mathbf{w}_\beta . \nabla \mathbf{v}_\beta - \nu_\beta \nabla^2 \mathbf{v}_\beta \right] = 0 \tag{13}$$

In the first and second parts of Eqs.(12)-(13), the *gravitational* versus the *inertial* contributions to the change in energy and momentum density of the field are apparent. Substitutions from (11) into (12)-(13) result in the invariant forms of conservation equations [6]

$$\frac{\partial \rho_\beta}{\partial t} + \mathbf{w}_\beta . \nabla \rho_\beta - D_\beta \nabla^2 \rho_\beta = \Omega_\beta \quad (14)$$

$$\frac{\partial T_\beta}{\partial t} + \mathbf{w}_\beta . \nabla T_\beta - \alpha_\beta \nabla^2 T_\beta = -h_\beta \Omega_\beta / (\rho_\beta c_{p\beta}) \quad (15)$$

$$\frac{\partial \mathbf{v}_\beta}{\partial t} + \mathbf{w}_\beta . \nabla \mathbf{v}_\beta - \nu_\beta \nabla^2 \mathbf{v}_\beta = -\mathbf{v}_\beta \Omega_\beta / \rho_\beta \quad (16)$$

An important feature of the modified equation of motion (16) is that it involves a convective velocity $\mathbf{w}_\beta$ that is different from the local fluid velocity $\mathbf{v}_\beta$. Because the convective velocity $\mathbf{w}_\beta$ is not *locally-defined* it cannot occur in *differential form* within the conservation equations [5]. This is because one cannot differentiate a function that is not locally, i.e. differentially, defined. To determine $\mathbf{w}_\beta$, one needs to go to the next higher scale (β+1) where $\mathbf{w}_\beta = \mathbf{v}_{\beta+1}$ becomes a local velocity. However, at this new scale one encounters yet another convective velocity $\mathbf{w}_{\beta+1}$ which is not known, requiring consideration of the higher scale (β+2). This unending chain constitutes the *closure problem* of the statistical theory of turbulence discussed earlier [5, 9].

3 Connection Between the Modified Form of the Equation of Motion and the Navier-Stokes Equation

The original form of the *Navier-Stokes* equation with constant coefficients is given as [1, 2]

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} . \nabla \mathbf{v} = -\nabla P + \mu \nabla^2 \mathbf{v} + \frac{1}{3} \mu \nabla (\nabla . \mathbf{v}) \quad (17)$$

Since thermodynamic pressure P_t is an isotropic scalar, P in (17) is not P_t. Rather, the pressure P is generally identified as the *mechanical pressure* that is defined in terms of the total stress tensor $T_{ij} = -P_t \delta_{ij} + \tau_{ij}$ as [7]

$$P_m = -(1/3) T_{ii} = P_t - (1/3) \tau_{ii} \quad (18)$$

The normal viscous stress is given by (10) as $(1/3)\tau_{ii} = (1/3)\rho \mathbf{v}_i \mathbf{V}_{ii} = -(1/3)\mu \nabla . \mathbf{v}$ and since $\nabla P_t \approx 0$ because of isotropic nature of P_t, the gradient of (18) becomes

$$\nabla P = \nabla P_m = \nabla \frac{1}{3} \mu (\nabla . \mathbf{v}) = \frac{1}{3} \mu \nabla (\nabla . \mathbf{v}) \quad (19)$$

Substituting from (19) into (17), the *Navier-Stokes* equation assumes the form

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} . \nabla \mathbf{v} - \nu \nabla^2 \mathbf{v} = 0 \quad (20)$$

that is almost identical to (16) with $\Omega_\beta = 0$ except that in (16) the convective velocity is different from the local velocity $\mathbf{v}_\beta$. However, because (20) includes a diffusion term and the $\mathbf{w}_\beta$ and $\mathbf{v}_\beta$ are related by $\mathbf{v}_\beta = \mathbf{w}_\beta + \mathbf{V}_\beta$, it is clear that (20) should in fact be written as (16).

4 Modified Theory of Laminar Boundary Layer Over a Flat Plate

As two examples of exact solutions of the modified equation of motion (16), the classical problems of two-dimensional and axi-symmetric jets [2] for laminar [8] and turbulent [9] flow were recently introduced. In this section, the solution of the modified equation of motion (16) for the classical problem of laminar flow within the boundary layer adjacent to a flat plate is considered. The convective velocity field (w'_x, w'_y) outside of the boundary layer, schematically shown in Fig.2, is known and given by [2]

$$w'_x = w'_o \qquad w'_y = 0 \quad (21)$$

Therefore, one looks for the *local* longitudinal velocity within the thin boundary layer adjacent to the wall at the scale β = c. For laminar cluster-dynamics [4, 5], the dissipative length scale is the cluster size $l_c \approx \lambda_m = 10^{-7}$ m i.e. that is the mean free-path of molecules. Also, a typical element size is $\lambda_c = L_m \approx 10^{-5}$ m. Finally, a typical system length is $L_c \approx 10^{-3}$ m that is about the thickness of the boundary layer. The relevant kinematic viscosity at LCD scale is estimated as [5]

$$\nu_c = l_c u_c / 3 = \lambda_m v_m / 3$$
$$\approx \frac{1}{3}(10^{-7}\, \mathrm{m} \times 300\ \mathrm{m/s}) = 0.1\ \mathrm{cm^2/s} \quad (22)$$

where v_m is mean molecular thermal speed. The notation (v'_x, v'_y) is chosen for the *local* axial and transverse velocities along the corresponding coordinates (x', y').

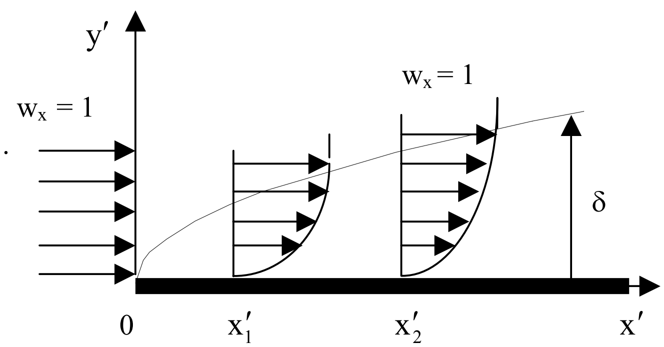

Fig.2 Laminar boundary layer over a flat plate.

The conventional boundary layer assumption $\partial^2/\partial x'^2 \ll \partial^2/\partial y'^2$ is introduced along with the dimensionless velocities

$$(\mathbf{v_x}, \mathbf{v_y}, \mathbf{w_x}, \mathbf{w_y}) = (\mathbf{v'_x}, \mathbf{v'_y}, \mathbf{w'_x}, \mathbf{w'_y})/\mathbf{w'_o} \qquad (23)$$

and coordinates

$$x = x'_e / l_H \quad , \quad y = y'/l_H \quad , \quad l_H = \nu / w'_o \qquad (24)$$

where l_H is the characteristic hydrodynamic length scale. The steady forms of (14) and (16) in the absence of chemical reactions $\Omega = 0$ reduce to

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \qquad (25)$$

$$w_x \frac{\partial v_x}{\partial x} + w_y \frac{\partial v_x}{\partial y} = \frac{\partial^2 v_x}{\partial y^2} \qquad (26)$$

that are subject to the boundary conditions

$$y = 0 \qquad v_x = v_y = 0 \qquad (27)$$

$$y \to \infty \qquad v_x = w_x = 1 \qquad (28)$$

Because usually $l_H = \nu / w'_o \ll 1$, the boundary layer coordinates (x, y) in (24) are stretched coordinates. The presence of boundary layer results in the transverse displacement of the outer flow field away from the plate.

According to (27)-(28), the local velocity v_x within the boundary layer must vanish at the plate and match the outer convective velocity field $w_x = 1$ at the edge of the boundary layer, i.e. in the limit $y \to \infty$ (Fig.2). Therefore, the convective velocity that is the mean of the local velocity $w_x = \langle v_x \rangle$ within the boundary layer will have the constant value of $w_x = ½$ at all axial locations. Introducing the value $w_x = ½$ and the similarity variable

$$\xi = \frac{y}{2\sqrt{2x}} \qquad (29)$$

into (26) and neglecting the transverse convection $w_y = 0$, one obtains

$$\frac{d^2 v_x}{d\xi^2} + 2\xi \frac{dv_x}{d\xi} = 0 \qquad (30)$$

that is subject to the boundary conditions

$$\xi = 0 \qquad v_x = 0 \qquad (31)$$

$$\xi \to \infty \qquad v_x = 1 \qquad (32)$$

The solution of (30)-(32) is

$$v_x = \mathrm{erf}(\xi) \qquad (33)$$

To facilitate the comparisons, the solution (33) is expressed as

$$v_x = \mathrm{erf}[\eta / 2\sqrt{2}] \qquad \eta = y/\sqrt{x} \qquad (34)$$

in terms of the same similarity variable η as in the classical theory [2, 10]. The boundary layer thickness is obtained from (33) as the position $\xi^* \simeq 1.8$ where $v_x = 0.99$ that by (29) leads to

$$\delta \simeq 5.1 x^{1/2} = 5.1\sqrt{Re_x} \qquad (35)$$

that is in close agreement with the classical numerical result of *Blasius* [2, 10]

$$\delta \simeq 5.0\sqrt{Re_x} \qquad (36)$$

One can express the solution (34) in terms of the stream function

$$\Psi = 2\sqrt{2x}\int_0^\xi \mathrm{erf}(\xi)d\xi \qquad (37)$$

The transverse velocity that is the solution of (25) and (27) is obtained from (37) as

$$v_y = \sqrt{\frac{2}{x}}\left[\xi \mathrm{erf}(\xi) - \int_0^\xi \mathrm{erf}(z)dz\right] \qquad (38)$$

Some of the streamlines calculated from (37) in terms of (x, y) coordinates using (29) are shown in Fig.3.

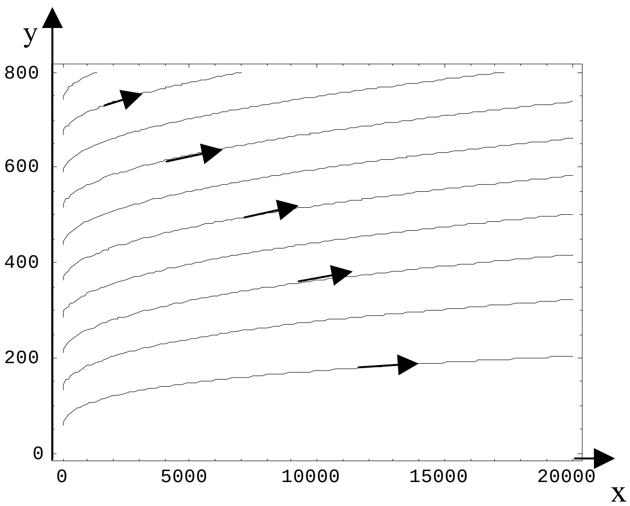

Fig.3 Calculated streamlines from (37) for laminar flow over a flat plate.

The predicted velocity profile calculated from (34) is shown in Fig.4a along with the experimental data of *Nikuradse* [11].

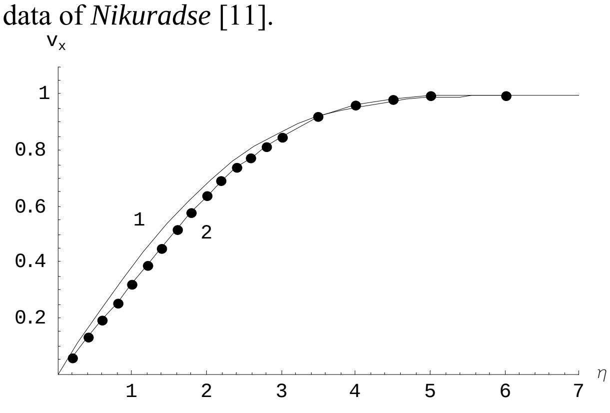

Fig.4a Comparisons between: (1) predicted axial velocity profile from (34), (2) the experimental data of Nikuradse [11] that also closely approximate the numerical calculation of Blasius [2, 10]

Because the experimental data of *Nikuradse* [11] very closely follow the numerical calculations of *Blasius* [10], according to the Fig.7.9 of *Schlichting* [2], the solid line labeled 2 in Fig.4a closely approximates the numerical solution of *Blasius* [10]. Therefore, Fig.4a also presents an indirect thus approximate comparison between the result of modified theory presented herein and the classical exact numerical solution of *Blasius* given in Fig.7.9 of *Schlichting* [2].

Comparison between the predicted velocity profile (34) and the relatively more recent, 1952 as compared to 1942, data obtained by *Dhawan* [12] is shown in Fig.4b. It is clear that the more recent experimental data of *Dhawan* [12] is in excellent agreement with the modified theory. The earlier experimental data of *Nikuradse* [11] shown in Fig.4a are found to always locate on the lower boundary of the more recent data shown in Fig.4b. Of course, in view of the inevitable experimental uncertainties as well as the uncertainties in the transport coefficients used in the theory, exact correspondence between the theory and the experiment should not be expected.

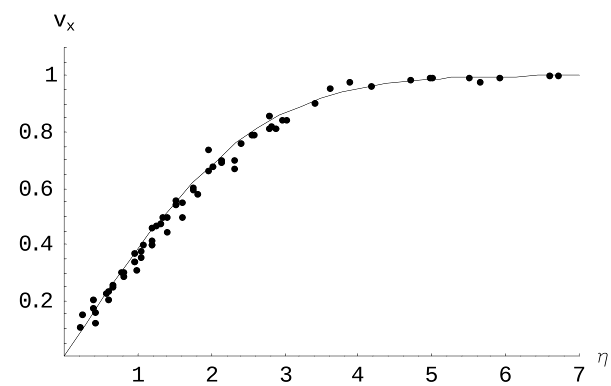

Fig.4b Comparison between the predicted axial velocity profile from (34) and the experimental data of Dhawan [12].

The magnitude of the transverse velocity at the edge of the boundary layer is obtained from (38) as

$$v_y(\infty) \simeq \sqrt{\frac{2}{\pi}}x^{-1/2} = 0.798\,\mathrm{Re}_x^{-1/2} \qquad (39)$$

that is in close agreement with the classical result of *Blasius* given as [2, 7]

$$v_y(\infty) = 0.861\,\mathrm{Re}_x^{-1/2} \qquad (40)$$

where the *Reynolds* number is $\mathrm{Re}_x = x = x'w'_o/\nu$.

Next, the predicted friction coefficients of the modified versus the classical theory are compared. The friction coefficient is defined as

$$C_f = \frac{-\tau}{\rho w_o'^2 / 2} \quad (41)$$

that involves the shear stress from (10)

$$\tau = -\mu \left(\frac{\partial v_x'}{\partial y'} \right)_{y'=0} \quad (42)$$

By substitutions from (23)-(24) in (42) and the use of (34) one obtains from (41)

$$C_f = \sqrt{\frac{2}{\pi}} x^{-1/2} = 0.798 \, \mathrm{Re}_x^{-1/2} \quad (43)$$

that is identical to (39). Indeed, the identity of (43) and (39) appears to be required by the energy balance $v_y(\infty)\ell t(\rho w_o'^2 / 2) = \tau \ell t$ over the length ℓ and the width t of the plate. The result (43) is in reasonable agreement with the classical result [2]

$$C_f = 0.664 \, \mathrm{Re}_x^{-1/2} \quad (44)$$

Experimental measurements of drag on a flat plate could be used to validate the predicted friction coefficient (43) of the modified theory.

6 Concluding Remarks

The modified form of the equation of motion was solved for the classical problem of *Blasius* concerning laminar incompressible flow in the boundary layer adjacent to a flat plate. The predicted velocity profile was found to be in excellent agreement with the more recent experimental data of *Dhawan* [12], and in close agreement with the earlier experimental data of *Nikuradse* [11] as well as the numerical solution of *Blasius* [10].

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