Proceedings of the 3rd IASME/WSEAS Int. Conf. on FLUID DYNAMICS & AERODYNAMICS, Corfu, Greece, August 20-22, 2005 (pp273-277)

An Analytical Approach to Turbulent Flow Through Smooth Pipes

AMIN MOOSAIE and GHOLAMALI ATEFI
Department of Mechanical Engineering
Iran University of Science and Technology
P.O. Box 16844, Narmak, Tehran
IRAN
aminmoosaie@mail.iust.ac.ir

Abstract: - In present paper the theory of the micropolar fluid based on a COSSERAT continuum model has been applied for analysis of turbulent flow through smooth pipes. The obtained results for the velocity field have been compared with known results from experiments done by Nikuradse in 1932. Nikuradse's experiment showed that the velocity profile in turbulent regime flattens in comparison with Navier-Stokes based solutions (i.e. Hagen-Poiseuille theory). After material identification, the results of solution have been shown for Reynolds numbers between 4×10^3 and 3.2×10^6 and the results demonstrate the wide range of validity of the solution.

Key-Words: - Turbulence-COSSERAT Continuum-Micropolar Fluid-Pipe Flow-Smooth Pipes

1 Introduction

The concept of COSSERAT continua was introduced in a paper submitted by two French brothers Cosserat [1] (also see Forest (2001)). In this continuum, we consider the effect of couples on a material element in addition to and independent of the effect of forces.

The theory of micro-fluids, introduced by Eringen [2,3], deals with fluids which exhibit certain microscopic effects arising from the local structure and micromotions of fluid element. A subclass of these is the micropolar fluid which has the microrotational effects and microrotational inertia. This theory is based on the concept of a COSSERAT continuum.

This class of fluids can support the couple stress, the body couples and the non-symmetric stress tensor and possess a rotation field, which is independent of the velocity field. The rotation field is no longer equal to the one half of the curl of velocity vector field. Because of the assumption of infinitesimal rotations, we can treat the rotation field as a vector field.

The theory, thus, has two independent kinematic variables; the velocity vector $\mathbf{V}$ and the spin or microrotation vector $\boldsymbol{\nu}$.

The linear constitutive equation for non-symmetric stress tensor (i.e. Cauchy's stress tensor), contains an additional viscosity coefficient k_v. The value of k_v shows the influence of the microrotation field on the stress tensor.

The linear constitutive equation for couple stress also contains three additional viscosity coefficients α_v, β_v and γ_v.

There are several approaches to the problem of turbulent flows. Some of them are models which constructed from experiments such as Boussinesq's model or k - ε model. But there are some analytical approaches by means of non-classical continuum models [4,5], for example the COSSERAT continua approach [3,6]. Another way to describe turbulence is the non-local models based on classical continuum models [7].

In present work we solve the problem of turbulent pipe flow by use of COSSERAT continuum mechanics. Then we compare our results with experiments done by Nikuradse.

The results of the experiment and also the results of our analysis are for the mean values of velocities defined by

$$\bar{\mathbf{V}}(\mathbf{x}) = \frac{1}{\Delta t} \int_{t_0}^{t_0+\Delta t} \mathbf{V}(\mathbf{x}, t)\, dt. \qquad (1)$$

It has been proposed by Eringen [3], Trostel [6] and other investigators to solve the problem of turbulent pipe flow by use of the theory of micropolar fluids. Here a material identification has been performed and the results have been compared with the results of Nikuradse [8,9].

2 Motion of a COSSERAT Fluid

2.1 Kinematics of COSSERAT Continua

At any material point of the continuum, we consider both a velocity and a rotation velocity vector denoted by $\mathbf{V}$ and $\boldsymbol{\nu}$, respectively. The so-called COSSERAT microrotation R_{ij} relates the current state of a triad of orthonormal directions attached to each material point to its initial state, i.e.

$$R_{ij} = \delta_{ij} - \Gamma_{ijk}\nu_k, \tag{2}$$

where δ_{ij} and Γ_{ijk} are the Kronecker delta tensor and permutation tensor, respectively.

The associated COSSERAT deformation ε_{ij} and torsion-curvature tensor κ_{ij} are written as

$$\varepsilon_{ij} = V_{j,i} - \Gamma_{ijk}\nu_k, \tag{3}$$

$$\kappa_{ij} = \nu_{j,i}, \tag{4}$$

where the comma denotes the partial differentiation.

2.2 Balance Laws in COSSERAT Media

It is assumed that the transfer of interaction between two particles of the continuum through a surface element $n_i ds$ occurs by means of both a traction vector $t_i ds$ and a moment vector $m_i ds$. Surface forces and couples are represented by the generally non-symmetric force-stress and couple-stress tensors t_{ij} and m_{ij}, respectively.

The axioms of balance of linear momentum and moment of momentum (i.e. angular momentum) require that the following equations hold

$$t_{ij,j} + f_i = \rho\frac{DV_i}{Dt}, \tag{5}$$

$$m_{ji,j} + \Gamma_{ijk}t_{ik} + l_i = j\frac{D\nu_i}{Dt}, \tag{6}$$

where ρ, j, f_i and l_i are the mass density, microinertia, body force per unit mass and body couple per unit mass respectively.

2.3 Constitutive Equations

Here we choose linear constitutive equations which describe our material behavior. It can be considered as the generalization of Newtonian fluids in the classical Navier-Stokes theory.

$$\begin{aligned} t_{kl} &= \left(-\pi + \lambda_v V_{r,r}\right)\delta_{kl} + \mu_v\left(V_{k,l} + V_{l,k}\right) \\ &\quad + k_v\left(V_{l,k} - \Gamma_{klr}\nu_r\right), \end{aligned} \tag{7}$$

$$m_{kl} = \alpha_v \nu_{r,r}\delta_{kl} + \beta_v \nu_{k,l} + \gamma_v \nu_{l,k}, \tag{8}$$

where π is the thermodynamic pressure.

As you see, the microrotation field has influence on the stress tensor, but the vice versa is not true.

2.4 Field Equations

At this stage we must mix the above equations to obtain governing field equations. The field equations for micropolar fluids in the vectorial form are given by

Conservation of mass (i.e. continuity equation)

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\left(\rho\mathbf{V}\right) = 0, \tag{9}$$

Balance of momentum

$$\begin{aligned} &\left(\lambda_v + 2\mu_v + k_v\right)\nabla\nabla\cdot\mathbf{V} - \left(\mu_v + k_v\right)\nabla\times\nabla\times\mathbf{V} \\ &+ k_v\nabla\times\boldsymbol{\nu} - \nabla\pi + \rho\left(\mathbf{f} - \frac{D\mathbf{V}}{Dt}\right) = \mathbf{0}, \end{aligned} \tag{10}$$

Balance of moment of momentum

$$\begin{aligned} &\left(\alpha_v + \beta_v + \gamma_v\right)\nabla\nabla\cdot\boldsymbol{\nu} - \gamma_v\nabla\times\nabla\times\boldsymbol{\nu} \\ &+ k_v\nabla\times\mathbf{V} - 2k_v\boldsymbol{\nu} + \rho\left(\mathbf{l} - j\frac{D\boldsymbol{\nu}}{Dt}\right) = \mathbf{0}, \end{aligned} \tag{11}$$

where $\frac{D}{Dt}$ denotes the material time derivative.

3 Definition of Problem

The flow through a pipe is determined by pressure gradient which is assumed to be constant in most of the problems with practical importance. The radius of pipe is R. The Oz axis overlaps the centerline of pipe. Because of symmetry, the upper half of the flow field is only considered. In this case the velocity components and the microrotation velocities become

$$V_r = V_\theta = 0, \quad V_z = u(r), \tag{12}$$

$$\nu_r = \nu_z = 0, \quad \nu_\theta = \nu(r). \tag{13}$$

The mass conservation law is identically satisfied with $\rho = \text{const}$. The equations of motion (i.e. Eq. (10) and Eq. (11)) are reduced to the following form by neglecting the body forces and body couples,

$$\left(\mu_v + k_v\right)\frac{d}{dr}\left(r\frac{du}{dr}\right) + k_v\frac{d}{dr}\left(r\nu\right) = r\frac{dp}{dz}, \tag{14}$$

$$\gamma_v\frac{d}{dr}\left(\frac{d\nu}{dr} + \frac{\nu}{r}\right) - k_v\frac{du}{dr} - 2k_v\nu = 0. \tag{15}$$

These two coupled ordinary differential equations must be subjected to the following boundary conditions

$$\text{At } r = 0 \quad : \quad \frac{du}{dr} = 0, \quad \frac{d\nu}{dy} = 0, \tag{16}$$

$$\text{At } r = R \quad : \quad u = 0, \quad \nu = -n\frac{du}{dr}. \tag{17}$$

4 Determination of Velocity Profile

The solutions of the Eq. (14) and Eq. (15), for the velocity and microrotation fields are

$$\frac{u(\sigma)}{U_0} = 1 - \sigma^2 + \frac{N\,k_v\,I_0(\xi)}{\xi(k_v + \mu_v)I_1(\xi)}\left[\frac{I_0(\xi\sigma)}{I_0(\xi)} - 1\right] \tag{18}$$

$$\frac{\nu R}{U_0} = \sigma - \frac{N I_1(\xi\sigma)}{I_1(\xi)}, \tag{19}$$

$$\xi = \sqrt{\frac{k_v\left(2\mu_v + k_v\right)}{\gamma_v\left(\mu_v + k_v\right)}} \tag{20}$$

$$\sigma = \frac{r}{R} \tag{21}$$

$$U_0 = -\tfrac{1}{2} R^2 (2\mu_v + k_v)^{-1}\left(\partial p/\partial z\right) \tag{22}$$

$$N = 1 - \frac{n(2\mu_v + k_v)}{(1-n)\mu_v k_v} \tag{23}$$

where I_0 and I_1 are modified Bessel functions of the first kind of order zero and one , respectively.

The process of validation of our results requires knowledge about material constants. Since the new material constants of COSSERAT theory is not known yet, we choose another way. We consider the flow field and the Nikuradse's results instead of boundary value problem. Thus we determine the coefficients from given data of experiment by means of an optimization process to minimize the fitting error. For this purpose and since the results of experiments have been reported for dimensionless velocities and radii, we apply following reconfiguration of the above solution.

$$\frac{U}{U_0} = 1 + N\frac{k_v}{k_v + \mu_v}\frac{1}{\xi}\frac{I_0(\xi)}{I_1(\xi)}\left[\frac{1}{I_0(\xi)} - 1\right], \tag{24}$$

where $U = u(\sigma = 0)$ is the velocity at centerline.

$$\frac{u(\sigma)}{U} = \frac{1 - \sigma^2 + N\dfrac{k_v}{k_v + \mu_v}\dfrac{1}{\xi}\dfrac{I_0(\xi)}{I_1(\xi)}\left[\dfrac{I_0(\xi\sigma)}{I_0(\xi)} - 1\right]}{1 + N\dfrac{k_v}{k_v + \mu_v}\dfrac{1}{\xi}\dfrac{I_0(\xi)}{I_1(\xi)}\left[\dfrac{1}{I_0(\xi)} - 1\right]} \tag{25}$$

Since the denominator of Eq. (25) is constant, we show the denominator by a single unknown constant and also mix the constants in the numerator to simplify the optimization process. Now, the coefficients of following velocity distribution must be determined from results of Nikuradse's experiments for smooth pipes.

$$\frac{u}{U} = C_1\left\{1 - \sigma^2 + C_2\left[\frac{I_0(\xi\sigma)}{I_0(\xi)} - 1\right]\right\} \tag{26}$$

At this stage, we determine the coefficients by minimization of fitting error for $Re = 4\times10^3$, $Re = 1.1\times10^5$ and $Re = 3.2\times10^6$ by use of an algorithm for nonlinear optimization like Levenberg-Marquardt and Evolution Strategies.

The results of COSSERAT and Navier-Stokes based solutions and Nikuradse's experiments have been shown in Figs. (1), (2) and (3) and the results can be compared.

5 Conclusion

In this paper we demonstrated that a complex phenomenon like turbulence could be modeled by use of a COSSERAT continuum model. A boundary condition was used for microrotation at walls to model the effect of surface quality on turbulence. It can be verified from this paper that continuum mechanical theories of higher orders such as COSSERAT model provide a way for analysis of turbulent flows in a very wide range of Reynolds numbers. The analysis of turbulent pipe flow by means of micropolar fluid mechanics has been proposed by Eringen [3], but no precise work has been done on this problem. The accuracy of our solution could be improved by considering slip boundary condition proved by Trostel [10] at pipe walls.

References:

[1] E. Cosserat and F. Cosserat, Théorie des Corps Déformables, *Hermann, A. et Fils*, 1909.

[2] C. A. Eringen, *Nonlinear Theory of Continuous Media*, McGraw-Hill Book Co., New York, 1962.

[3] C. A. Eringen, *Microcontinuum Field Theories II : Fluent Media*, Springer-Verlag, New York, 2001.

[4] C. Truesdell and R. Toupin, Classical field theories, *Handbuch der Physik*, Vol. III/1, Springer-Verlag, Berlin, 1960.

[5] C. Truesdell and W. Noll, The non-linear field theories of mechanics, *Handbuch der Physik*, Vol. III/3, Springer-Verlag, Berlin, 1965.

[6] R. Trostel, *Gedanken zur konstruktion mechanischer Theorien II*, Technische Universität Berlin-Forschungsberichte Nr. 7, Berlin, 1988.

[7] G. Silber, Nonlocal theories and modeling of fully developed turbulence, *Mechanics Research Communications*, **15** (4), Pergamon Press, 1988, pp. 199-203.

[8] J. Nikuradse, Gesetzmässigkeit der turbulenten Strömung in glatten Rohren, *Forschg. Arb. Ing.-Wes.*, No. 320, 1932.

[9] H. Schlichting, *Grenzschichttheorie*, Braun, Karlsruhe, 1964.

[10] Atefi Gh., Quer angeströmter drehender Zylinder bei kleinen Reynoldszahlen und bei Schlupf, *Archive of Applied Mechanics*, **61**, 1991.

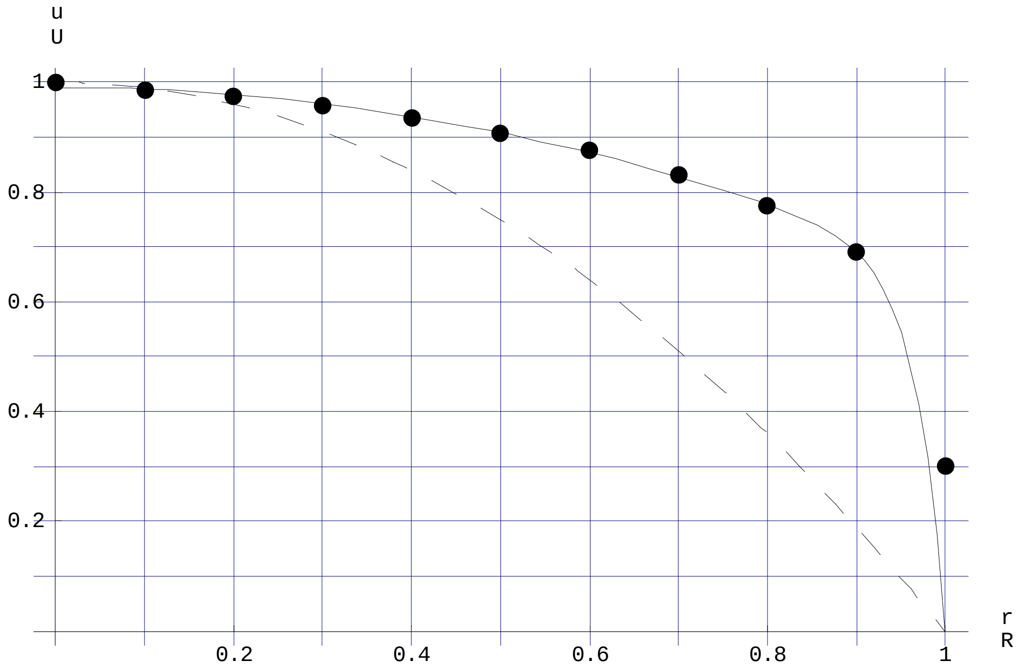

Fig. 1 Velocity Profile for $\mathrm{Re} = 4 \times 10^3$

Solid line : COSSERAT Dashed line : Navier-Stokes Points : Experiment

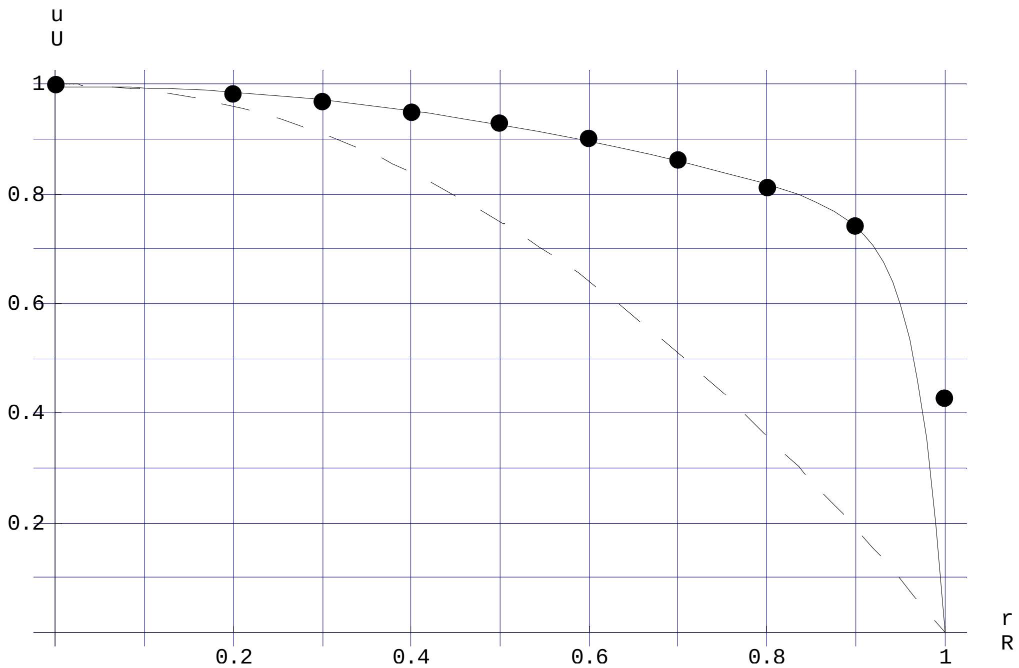

Fig. 2 Velocity Profile for $\mathrm{Re} = 1.1 \times 10^5$

Solid line : COSSERAT Dashed line : Navier-Stokes Points : Experiment

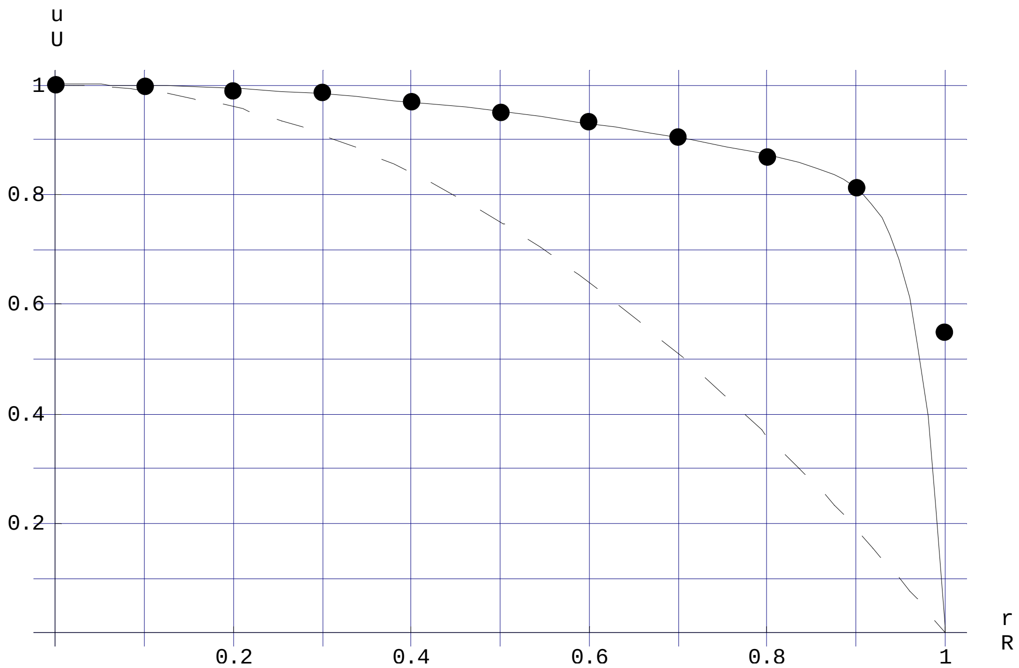

Fig. 3 Velocity Profile for $Re = 3.2 \times 10^6$

Solid line : COSSERAT Dashed line : Navier-Stokes Points : Experiment