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Stability properties of a family of chock capturing methods for hyperbolic conservation laws

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Abstract: In this paper a family of local shock capturing methods for hyperbolic conservation laws is presented. We analyze some stability properties in order to obtain the convergence of the schemes.

Key Words: Hyperbolic conservation laws, local shock capturing methods, stability, convergence.

1 Introduction

We consider numerical approximations to weak solutions of nonlinear hyperbolic conservation laws:

$$u_t + f(u)_x = 0, \tag{1}$$

$$u(x,0) = u_0(x). \tag{2}$$

the initial data $u_0(x)$ are supposed to be piecewise smooth functions that either periodic or of compact support.

Let be $u_j^n = u_h(x_j, t_n)$ denote a numerical approximation to the exact solution $u(x_j, t_n)$ of (1)-(2) defined on a computational grid $x_j = jh$, $t_n = n\Delta t$ in conservation form:

$$u_j^{n+1} = u_j^n - \lambda(\hat{f}^n_{j+\frac{1}{2}} - \hat{f}^n_{j-\frac{1}{2}}), \tag{3}$$

where $\lambda = \frac{\Delta t}{h}$ and the numerical flux is a function of 2k variables

$$\hat{f}^n_{j+\frac{1}{2}} = \hat{f}(u^n_{j-k+1}, \ldots, u^n_{j+k}), \tag{4}$$

which is consistent with (1), i.e.

$$\hat{f}(u, \ldots, u) = f(u). \tag{5}$$

The importance of the following lemma (see [5]) is because it implies that approximating the numerical flux $\hat{f}_{j+\frac{1}{2}}$ to a high order accuracy it is enough to reconstruct $g(x_{j+\frac{1}{2}})$ (see equation (6)) up to the same order.

Lemma 1 (Shu and Osher) *If a function $g(x)$ satisfies*

$$f(u(x)) = \frac{1}{h}\int_{x-\frac{h}{2}}^{x+\frac{h}{2}} g(\xi)d\xi \tag{6}$$

then

$$f(u(x))_x = \frac{g(x+\frac{h}{2}) - g(x-\frac{h}{2})}{h}.$$

First order methods give poor accuracy in smooth regions of the flow. Shocks tend to be heavily smeared and poorly resolved on the grid. These effects are due to the large amount of numerical dissipation in these schemes.

Marquina [4] introduced a new local third order accurate shock capturing method (PHM), the main advantage of this method lies on the property that it is localer than ENO and TVD upwind schemes of the same order, (and, thus, giving better resolution of corners), because numerical fluxes depend only on four variables.

In [1] and [2] we present some modifications of the PHM. They are based on different reconstructions. From the numerical experiments, the methods become efficient since they are low cost and they are not very sensitive to the Courant-Friedrichs-Lewy (CFL) number.

To complete the schemes, Shu and Osher developed a special family of Runge-Kutta time integration schemes that have a TVD property [5], [6]. The TVD property prevents the time stepping scheme from introducing spurious spatial oscillations into upwind-biased spatial discretization.

The aim of this paper is to analyze some stability properties in order to obtain the convergence of these type of schemes. The paper is organized as follows: section 2 contains the reconstruction step. In section 3 the complete algorithm is presented. Finally, some stability properties and a convergence analysis are studied in 4.

2 Piecewise reconstructions

We define a computational grid $x_j = jh$, j integer, $h > 0$, where the cells are

$$C_j = \{x : x_{j-\frac{1}{2}} \leq x \leq x_{j+\frac{1}{2}}\}, \qquad (7)$$

where $x_{j+\frac{1}{2}} = x_j + \frac{1}{2}h$.

Our grid data are:

$$v_j = \frac{1}{h}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} g(\xi)d\xi, \qquad (8)$$

$$d_{j+\frac{1}{2}} = \frac{v_{j+1} - v_j}{h}, \qquad (9)$$

$(d_{j+\frac{1}{2}} = g^{'}(x_{j+\frac{1}{2}}) + O(h^2))$.

We required the following conditions for every j:

$$v_j = \frac{1}{h}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} r_j(\xi)d\xi, \qquad (10)$$

$$d_{i+\frac{1}{2}} = r_j^{'}(x_{i+\frac{1}{2}}),\ i = j-1, j. \qquad (11)$$

Taylor series expansions show that conditions (10) and (11) imply third order accuracy of the reconstruction $r_j(x)$.

2.1 Local Total Variation Bounded reconstructions

The reconstruction procedure is repeated at every time step, thus the change in total variation of the reconstruction must be controlled. The local total variation of the reconstruction $\{r_j\}$ is defined by

$$LTV_j = TV(r_j),$$

where $TV(r_j)$ means the total variation of the function $r_j(x)$ in the cell C_j. The size of LTV_j determines locally the increasing of the total variation of the reconstruction. In [4], the following definition is given:

Definition 1 *A method of reconstruction is local total variation bounded (LTVB) if there exists a constant $M > 0$, independing of h (depending only on the function $g(x)$ to be reconstructed), such that*

$$LTV_j \leq M \cdot h, \quad for\ all\ j.$$

The method of reconstruction with $r_j(x)$ is not LTVB. For nonlinear fluxes, the method becomes "unstable". As PHM scheme, we will assign a different value to the central derivative.

Since our reconstruction is local, we restrict our discussion to the cell C_0 and the grid data of the cell: $v_0, d_{-\frac{1}{2}}$, and $d_{\frac{1}{2}}$.

The algorithm defines the modified reconstruction $\tilde{r}_0(x)$ such that its derivative interpolates d_0 at x_0 and the lateral grid derivative with smallest absolute value. Where, if C_0 is a nontransition ($d_{-\frac{1}{2}} \cdot d_{\frac{1}{2}} > 0$) cell, then we define d_0 such that $|d_0 - r'(x_0)| = O(h^2)$ in smooth regions and $\max(\tilde{r}_0(x_{-\frac{1}{2}}), \tilde{r}_0(x_{\frac{1}{2}})) = O(1)$. In transition cells we consider $d_0 = 0$.

Theorem 1 *The above method of reconstruction of the function $g(x)$ is a local preprocessed reconstruction procedure that is LTVB.*

3 Piecewise Methods

The final algorithm is based on the first order Roe scheme, with the entropy-fix correction due to Shu and Osher [6], for local piecewise reconstruction.

The numerical fluxes are reconstructed from the upwind side, except that if the wind changes direction at the cell, then a local Lax-Friedrichs flux decomposition is performed.

ALGORITHM

For every j **do**

If $f'(u)$ does not changes of sign between u_j^n and u_{j+1}^n, then

Upwindness Phase

$$\bar{a}_{j+\frac{1}{2}} = \frac{v_{j+1} - v_j}{u_{j+1} - u_j} \tag{12}$$

(Roe-speed)

If $\bar{a}_{j+\frac{1}{2}} \geq 0$ then

$$\hat{f}_{j+\frac{1}{2}} = \tilde{r}_j(x_{j+\frac{1}{2}}) \tag{13}$$

else

$$\hat{f}_{j+\frac{1}{2}} = \tilde{r}_{j+1}(x_{(j+1)-\frac{1}{2}}) \tag{14}$$

else

Flux decomposition phase

$M_{j+\frac{1}{2}} = max_{u_j^n < u < u_{j+1}^n} |f'(u)|$

$v_k^+ = \frac{1}{2}(v_k + M_{j+\frac{1}{2}} u_k^n)$ $k = j-1, j, j+1$

Computation of f^+, as (13)

$v_k^- = \frac{1}{2}(v_k - M_{j+\frac{1}{2}} u_k^n)$ $k = j, j+1, j+2$

Computation of f^-, as (14)

$\hat{f}_{j+\frac{1}{2}} = f^+ + f^-$

end

4 Stability and Convergence Analysis

Our method is consistent and in conservation form, then the Lax-Wendroff theorem ([3] Chapter 12) says that if a sequence of the approximations converges then the limit is a weak solution.

Definition 2 *The Total Variation of a discrete solution is defined by*

$$TV(u) = \sum_j |u^n_{j+1} - u^n_j|$$

To obtain the convergence, we use the following two theorems (Theorem 15.1 and Theorem 15.2 in [3]).

Theorem 2 *Consider a conservative method with a Lipschitz continuous numerical flux and suppose that for each initial data u_0 there exists some $k_0, R > 0$ such that*

$$TV(u^n) \leq R \quad \forall n, k \quad \text{with } \mathrm{k} < \mathrm{k}_0,\ \mathrm{nk} \leq \mathrm{T}.$$

Then the method is TV-stable.

Theorem 3 *A conservative scheme in conservation form, with Lipschitz continuous numerical flux and TV-stable is convergent.*

Thus, it is enough to guarantee that the numerical flux is Lipschitz continuous and that the $TV(u^n)$ is uniformly bounded for all n, k with $k < k_0$, $nk \leq T$.

We assume that the numerical flux associated to the considered reconstruction is Lipschitz continuous.

Finally, we have to find a positive constant C such that $TV(u^n) < C$ for all n, k with $k < k_0$, $nk \leq T$. In practice, a Runge-Kutta TVD for the time discretization is used. Thus, we do not have stability problems with the time. In order to simplify the notation we consider the simplest Euler scheme. From

$$\begin{aligned} u^{n+1}_j &= u^n_j - \lambda(\hat{f}^n_{j+\frac{1}{2}} - \hat{f}^n_{j-\frac{1}{2}}), \\ u^{n+1}_{j-1} &= u^n_{j-1} - \lambda(\hat{f}^n_{j-\frac{1}{2}} - \hat{f}^n_{j-\frac{3}{2}}). \end{aligned}$$

we obtain

$$(u^{n+1}_j - u^{n+1}_{j-1}) = (u^n_j - u^n_{j-1}) - \lambda(\hat{f}_{j+\frac{1}{2}} - 2\hat{f}_{j-\frac{1}{2}} + \hat{f}_{j-\frac{3}{2}}). \tag{15}$$

Remark 1 *We are interesting to analyze the reconstruction step only. The others parts of the algorithm is classical and common for this type of schemes. Thus, in order to present a more clear proof, we do not consider the* upwind phase.

We summarize the basic ingredients to obtain the desired stability.

1) Let us choose a number $h > 0$, such that there is at least two cells between two jumps of $g(x)$. Since $g(x)$ is a piecewise smooth function, there exist two constant $M_1, M_2 > 0$ depending only on derivatives of g in smooth regions, such that:

a) For all j, except for a finite number of insolated $j's$ (for which $d_{j+\frac{1}{2}} = O(h^{-1})$), $|d_{j+\frac{1}{2}}| < M_1$. In particular, d_c and d_l are lees than M_1 always.

b)

$$h|d_{j+\frac{1}{2}} - r'(x_j)| \leq M_2(|u^n_{j+1} - u^n_j| + |u^n_j - u^n_{j-1}|).$$

c)

$$h(r'(x_j) - r'(x_{j-1})) = f'(\beta^n_j)(u^n_j - u^n_{j-1}).$$

2) If let us consider $v^n_j = u^n_{j-1}, \forall j$. Since f is smooth, $f(u^n_j) - f(v^n_j) = f'(\theta^n_j)(u^n_j - v^n_j)$.

3) Let be the CFL condition $0 \leq \alpha^n_j := \lambda(f'(\theta^n_j) + \frac{f'(\beta^n_j)}{2}) \leq 1$ for all j.

Using this properties and relation (15) we arrive to

$$|u^{n+1}_j - u^{n+1}_{j-1}| \quad \leq \quad (1 - \alpha^n_j)|u^n_j - u^n_{j-1}|$$

$$
\begin{aligned}
+ \quad & \alpha_{j-1}^n |u_{j-1}^n - u_{j-2}^n| \\
+ \quad & \Delta t M_2 (|u_{j+1}^n - u_j^n| \\
& +2|u_j^n - u_{j-1}^n| \\
& +|u_{j-1}^n - u_{j-2}^n|) \frac{1}{12},
\end{aligned}
$$

and then

$$TV(u^{n+1}) \leq (1 + \alpha k) TV(u^n),$$

where $0 \leq \alpha \leq \frac{1}{3} M_2$ is independent of u^n.

By induction

$$TV(u^{n+1}) \leq (1 + \alpha k)^{n+1} TV(u^0).$$

Thus, for $(n+1)k \leq T$,

$$TV(u^{n+1}) \leq e^{\alpha T} TV(u^0),$$

and hence the method is total variation stable.

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