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Complex Ginzburg-Landau Equation for Suddenly Blocked Unsteady Channel Flow

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Abstract: - Spatio-temporal dynamics of complex flows in fluid mechanics is often analyzed and modeled by means of relatively simple nonlinear evolution equations, for instance, the complex Ginzburg-Landau equation. Such equations are much simpler than the original nonlinear Navier-Stokes equations but in many cases essential characteristics of the flow can be recovered from the model equation. Weakly nonlinear theory is used to derive an amplitude evolution equation for the most unstable mode for a flow between two parallel infinite planes. The base flow is subjected to rapid instantaneous deceleration so that the total fluid flux through the cross-section of the channel is equal to zero. The quasi-steady assumption is used in the derivation, that is, it is assumed that the rate of change of the base flow with respect to time is much smaller than the growth rate of perturbations. It is shown that the evolution equation is the complex Ginzburg-Landau equation.

Key-Words: - weakly nonlinear theory, Ginzburg-Landau equation

1 Introduction

There are many examples in fluid mechanics where relatively simple amplitude evolution equations are used to describe spatio-temporal dynamics of complex flows. It is shown in [1] that the transition in a wake behind a cylinder, in an interval of the Reynolds numbers around the critical value, is described by the Landau equation. The Ginzburg-Landau (GL) equation is used in [2] to analyze the dynamics of the flow behind bluff rings. The coefficients of the GL equation are computed from the experimental data. Good quantitative agreement is found between experimental data and the results of numerical modeling. Different flow patterns created by the wakes of a row of 16 circular cylinders placed close to each other in an incoming flow are analyzed in [3]. Spatio-temporal characteristics of the flow are recorded and the data are used to evaluate the coefficients of the GL equation. The validity of the model is assessed by reproducing experimentally observed flow patterns from the model.

In these three cases the GL equation (or Landau equation) was assumed and used as a model equation. In the present paper we derive the GL equation from the Navier-Stokes equations for an important practical case of rapidly decelerated laminar flow in a plane channel. Rapidly changing unsteady flows are important in many applications. Typical examples include the design and analysis of water supply systems, flows in natural gas pipelines and blood flow in arteries. Pump shutdowns and rapid changes in valve settings are known to generate unsteady flows in hydraulic devices or water supply systems. The pressure in unsteady flow can cause cavitation, pitting and corrosion [4]. An atherosclerosis plaque development in regions where shear stress changes direction can be triggered by unsteady blood flow [5]. In all these examples the flow characteristics (velocity and pressure) change considerably during short time intervals as a result of rapid deceleration or acceleration of the flow. In order to avoid possible problems associated with short time transients one needs to have an efficient model for

calculating flow characteristics in a plane channel or a pipe.

Consider a fully developed Poiseuille flow in a plane channel between two infinite parallel plates $\tilde{y} = -h$ and $\tilde{y} = h$. Starting from time $\tilde{t} = 0$ the flow is suddenly blocked so that the total fluid flux through the cross-section of the channel is equal to zero for $\tilde{t} > 0$. Assuming that the velocity vector has only one non-zero component, $\tilde{U}(\tilde{y}, \tilde{t})$, which depends only on the transverse coordinate $\tilde{y}$ and time $\tilde{t}$, we obtain the following equation

$$\frac{\partial \tilde{U}}{\partial \tilde{t}} = -\frac{1}{\rho}\frac{\partial \tilde{P}}{\partial \tilde{x}} + \nu \frac{\partial^2 \tilde{U}}{\partial \tilde{y}^2}, \qquad (1)$$

where ρ and ν are the density and the kinematic viscosity of the fluid, respectively, $\tilde{P}$ is the pressure and $\tilde{x}$ is the longitudinal coordinate. The function $\tilde{U}$ satisfies the following boundary and initial conditions:

$$\tilde{U}|_{\tilde{y}=\pm h} = 0, \quad \tilde{U}|_{\tilde{t}=0} = \frac{1}{2\rho\nu}\frac{d\tilde{P}_0}{d\tilde{x}}(\tilde{y}^2 - h^2), \qquad (2)$$

where $d\tilde{P}_0 / d\tilde{x}$ is a constant pressure gradient in the $\tilde{x}$ direction before deceleration. The total fluid flux through the cross-section of the channel is zero for all $\tilde{t} > 0$:

$$\int_{-h}^{h} \tilde{U}(\tilde{y}, \tilde{t})\, d\tilde{y} = 0. \qquad (3)$$

Problem (1) – (3) is solved by a Pohlhausen type technique in [6] and by the method of matched asymptotic expansions in [7]. The velocity profiles are found to contain inflection points and are therefore potentially unstable for small perturbations. The linear stability analysis used in [8] is based on a quasi-steady assumption which implies that the base flow velocity profiles are "frozen" so that the time variable for the base flow is considered as a parameter. The validity of the quasi-steady assumption is assessed in [9] where full linearized disturbance equations are solved numerically as an initial value problem. Numerical results in [9] show that the growth rates of perturbations for oscillatory pipe flows deviate considerably from the results of the quasi-steady approach. On the other hand, it is shown in [9] that the results from the quasi-steady theory are in good agreement with the initial value problem approach for the case of rapidly decelerated channel and pipe flows. Linear stability analysis in [8] shows that the flow $\tilde{U}$ is unstable in a wide range of Reynolds numbers. Critical values of the Reynolds number, wavenumber and wave speed are calculated in [8].

Linear stability analysis is used to describe the onset of instability in a fluid system. It provides a marginal stability curve and the critical values of the parameters at the threshold. However, the evolution of the most unstable mode cannot be predicted by a linear theory. There are several ways to analyze the evolution of an unstable perturbation analytically. One widely used approach includes the analysis of model amplitude evolution equations (for example, the complex Ginzburg-Landau equation, see [10] – [12]). These equations are not derived from the Navier-Stokes equations but are rather used as a phenomenological model for the flow of interest. In some cases the coefficients of the equations are determined from experimental data. Another option is to consider a Reynolds number which is slightly above the critical value. If one restricts attention to the neighborhood of the critical point $(\mathrm{Re}_c, \alpha_c, c_c)$, where Re is the Reynolds number, α is the wavenumber, c is the wave speed and the subscript c indicates the critical values of the parameters, then the growth rates of a perturbation will be small in the vicinity of the critical point. This allows one to use the methods of weakly nonlinear theory and derive an amplitude evolution equation for the most unstable mode, assuming that the amplitude is a slowly varying function of the longitudinal coordinate and time. This approach is used, for example, in [13] for the case of a plane Poiseuille flow. The weakly nonlinear analysis of a problem related to the generation of waves by wind in performed in [14]. Recently an amplitude evolution equation for the most unstable mode was derived in [15] for the case of a rapidly decelerated flow in a pipe. Two examples of weakly nonlinear analysis for shallow flows can be found in [16] and [17]. In all the above mentioned cases the amplitude evolution equation is found to be the complex Ginzburg-Landau equation.

In the present paper a complex Ginzburg-Landau equation is derived under the quasi-steady assumption for the flow between two parallel infinite plates. The base flow in this case is unsteady and is given as the solution of (1) – (3). The linear stability calculations presented in [8] show that the flow (1) – (3) is linearly unstable for a certain range of the Reynolds numbers. Assuming that Re is slightly larger than the critical value Re_c and using the methods of weakly nonlinear theory we derive an amplitude evolution equation for the most unstable mode. It is shown that the evolution equation is the complex Ginzburg-Landau equation whose coefficients depend on the solution of the linearized stability problem.

2 Basic properties of the Ginzburg-Landau equation

It is found that in many applied hydrodynamic problems the dynamics of the flow above the threshold can be described by a complex Ginzburg-Landau equation of the form

$$\frac{\partial A}{\partial \tau} = \sigma A + \delta \frac{\partial^2 A}{\partial \xi^2} + \mu |A|^2 A, \qquad (4)$$

where $\sigma = \sigma_r + i\sigma_i$, $\delta = \delta_r + i\delta_i$, $\mu = \mu_r + i\mu_i$ are complex coefficients. It is known that, depending on the values of the coefficients, equation (4) possesses a rich variety of solutions, including some solutions which may even look like chaotic solutions. An excellent review of the properties of the Ginzburg-Landau equation (4) is given in [11]. The right-hand side of equation (4) contains terms which are related to linear amplification, diffusion and nonlinear saturation. The coefficients of (4) have the following physical meaning. The real part of σ, namely, σ_r, represents the rate of amplification of an unstable perturbation. The angular frequency of the oscillation is given by σ_i. The dependence of the instability growth rate and oscillation frequency on the wavelength is reflected by the coefficients δ_r and δ_i, respectively. If $\mu_r < 0$, the nonlinearities tend to saturate the instability. From a physical point of view, this means that there exists another equilibrium state after the flow loses stability. Examples of such equilibrium states are the Rayleigh-Benard convection between two parallel plates which are maintained at different constant temperatures and the Taylor-Couette flow between two rotating cylinders (see [18]). Such a situation is referred to as "supercritical instability" in the hydrodynamic stability literature.

On the other hand, if $\mu_r > 0$, then higher order terms on the right-hand side of (4) are also important and (4) is much less informative. Such a case is known as "subcritical instability". One example of subcritical instability is given in [13] for the case of a plane Poiseuille flow. Note that μ_r in equation (4) is usually referred to as the Landau constant in the literature.

3 Derivation of the Ginzburg-Landau equation

A two-dimensional incompressible viscous flow in a plane channel can be described by the following dimensionless equation

$$(\Delta\psi)_t + \psi_y(\Delta\psi)_x - \psi_x(\Delta\psi)_y = \frac{1}{\mathrm{Re}}\Delta^2\psi, \quad (5)$$

where $\mathrm{Re} = U_{\max}h/\nu$, $U_{\max}$ is the maximum velocity of the undisturbed flow and $\psi(x,y)$ is the stream function defined by the relations $u = \psi_y$, $v = -\psi_x$. Consider a perturbed solution to (5) in the form

$$\psi = \psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2 + \ldots \quad (6)$$

where ε is a small parameter and $\psi_{0y} = U$ is the dimensionless solution to (1) – (3). The solution $U(y,t)$ can be found by the method of the Laplace transform and has the form

$$U(y,t) = -\frac{4}{3}\sum_{n=1}^{\infty}\frac{\beta_n\cos(\beta_n y) - \sin\beta_n}{\beta_n^2\sin\beta_n}e^{-\beta_n^2 t} \quad (7)$$

where β_n are the roots of the equation $\tan\beta_n = \beta_n$.

Substituting (6) into (5) and keeping only linear terms with respect to ε, we obtain the following equation which governs the linear stability of the flow:

$$L\psi_1 = 0, \quad (8)$$

where

$$L\eta = \eta_{xxt} + \eta_{yyt} + U\eta_{xxx} + U\eta_{yyx} - U_{yy}\eta_x - \mathrm{Re}^{-1}(\eta_{xxxx} + 2\eta_{xxyy} + \eta_{yyyy}) \quad (9)$$

Using the method of normal modes and assuming the solution to (8) to be in the form

$$\psi_1(x,y,t) = \varphi_1(y)\exp[i\alpha(x-ct)] + c.c. \quad (10)$$

where α is the wavenumber, c is the wave speed of a perturbation, and the commonly used fluid dynamics short-hand notation $c.c.$ means the "complex conjugate" of the first term on the right-hand side of the equation where it appears, we obtain the Orr-Sommerfeld equation in the form

$$\varphi_1^{////} - (2\alpha^2 + i\alpha U\,\mathrm{Re})\varphi_1^{//} + (i\alpha^3 U\,\mathrm{Re} + i\alpha\,\mathrm{Re}U_{yy} + \alpha^4)\varphi_1 = -i\alpha\,\mathrm{Re}\,c(\varphi_1^{//} - \alpha^2\varphi_1). \quad (11)$$

The boundary conditions are

$$\varphi_1(\pm 1) = 0, \quad \varphi_1^{/}(\pm 1) = 0. \quad (12)$$

The boundary value problem (11) – (12) is an eigenvalue problem which determines the critical values of the parameters Re, α and c. Recall that the base flow velocity U is a function of y and t, but we adopted here the quasi-steady assumption. Thus, t is considered as a parameter in $U = U(y,t)$. It is shown in [15] that the quasi-steady assumption is justified if the rate of change of the base flow velocity with respect to time is smaller than the growth rate of the perturbation. Numerical results presented in [15] indicate that the quasi-steady assumption is appropriate for the case of suddenly blocked pipe flows. In addition, numerical solution of full linearized disturbance equations (solved as an initial value problem) in [9] showed that the calculated growth rates are in good agreement with the quasi-steady theory. Therefore, the quasi-steady assumption is adopted in the present study.

The critical Reynolds numbers are calculated in [8] by means of a numerical solution of (11) – (12). The structure of the most unstable mode can also be obtained from the solution of (11) – (12) but the linear theory cannot be used to estimate the amplitude of the most unstable mode and to describe the evolution of the most unstable mode. The next natural step is to use the methods of weakly nonlinear theory (see, for example, [13]) in order to derive the amplitude evolution equation for the case where Re is slightly larger than the critical value Re_c. Following [13] we assume that

$$\mathrm{Re} = \mathrm{Re}_c(1+\varepsilon^2) \tag{13}$$

and introduce the slow time τ and the stretched longitudinal variable ξ such that

$$\tau = \varepsilon^2 t\,,\quad \xi = \varepsilon(x - c_g t), \tag{14}$$

where c_g is the group velocity.

The differential operators $\partial/\partial t$ and $\partial/\partial x$ are then replaced by

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} - \varepsilon c_g \frac{\partial}{\partial \xi} + \varepsilon^2 \frac{\partial}{\partial \tau},$$
$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \varepsilon \frac{\partial}{\partial \xi}. \tag{15}$$

The function ψ_1 in (6) is represented in the form

$$\psi_1 = A(\xi,\tau)\varphi_1(y)\exp[i\alpha_c(x - c_c t)] + c.c. \tag{16}$$

where $A(\xi,\tau)$ is a slowly varying amplitude and $\varphi_1(y)$ is an eigenfunction of the linear stability problem (11) – (12) at $\mathrm{Re} = \mathrm{Re}_c$, $\alpha = \alpha_c$ and $c = c_c$. The evolution equation for A is obtained by taking higher terms of the perturbation expansion (6) into account. Substituting (13) – (16) into (6) and collecting the terms of order ε^2 we obtain

$$L\psi_2 = c_g(\psi_{1xx\xi} + \psi_{1yy\xi}) - 2\psi_{1x\xi t} - U(3\psi_{1xx\xi} + \psi_{1yy\xi}) - \psi_{1y}(\psi_{1xxx} + \psi_{1yyy}) + \psi_{1x}(\psi_{1xxy} + \psi_{1yyy}) + U_{yy}\psi_{1\xi} - \mathrm{Re}_c^{-1}U_{yyy} + 4\,\mathrm{Re}_c^{-1}(\psi_{1xxx\xi} + \psi_{1xyy\xi}) \tag{17}$$

Similarly, substituting (13) – (16) into (6) and collecting the terms of order ε^3 we get

$$L\psi_3 = c_g(\psi_{2xx\xi} + \psi_{2yy\xi} + 2\psi_{1x\xi\xi}) - \psi_{1xx\tau} - \psi_{1yy\tau} - \psi_{1\xi\xi t} - U(3\psi_{2xx\xi} + \psi_{2yy\xi} - 3\psi_{1x\xi\xi}) - \psi_{1y}(\psi_{2xxx} + \psi_{2yyx}) - \psi_{1y}(3\psi_{1xx\xi} + \psi_{1yy\xi}) + \psi_{1\xi}(\psi_{1xxy} + \psi_{1yyy}) + U_{yy}\psi_{2\xi} - \psi_{2y}(\psi_{1xxx} + \psi_{1yyx}) + \psi_{2x}(\psi_{1xxy} + \psi_{1yyy}) + \psi_{1x}(\psi_{2xxy} + \psi_{2yyy}) + 2\psi_{1x}\psi_{1xy\xi} - \mathrm{Re}_c^{-1}(\psi_{1xxxx} + 2\psi_{1xxyy} + \psi_{1yyyy}) + \mathrm{Re}_c^{-1}(4\psi_{2xxx\xi} + 4\psi_{2xyy\xi} + 6\psi_{1xx\xi\xi} + 2\psi_{1yy\xi\xi}). \tag{18}$$

The solution to (17) is sought in the form

$$\psi_2 = A^2\varphi_2^{(0)}(y)\exp[2i\alpha_c(x - c_c t)] + AA^*\varphi_2^{(1)}(y) + A_\xi\varphi_2^{(2)}(y)\exp[i\alpha_c(x - c_c t)] + c.c. \tag{19}$$

where A^* is the complex conjugate of A. Substituting (19) and (16) into (17) and collecting the terms proportional to $A^2\exp[2i\alpha_c(x - c_c t)]$ we obtain the following equation for the function $\varphi_2^{(0)}(y)$:

$$-\mathrm{Re}_c^{-1}(\varphi_{2yyyy}^{(0)} - 8\alpha_c^2\varphi_{2yy}^{(0)} + 16\alpha_c^4\varphi_2^{(0)}) + 2i\alpha_c\varphi_{2yy}^{(0)}(U - c_c) - 8i\alpha_c^3(U - c_c)\varphi_2^{(0)} - 2i\alpha_c U_{yy}\varphi_2^{(0)} = -i\alpha_c\varphi_{1y}\varphi_{1yy} + i\alpha_c\varphi_1\varphi_{1yyy} \tag{20}$$

with the boundary conditions

$$\varphi_2^{(0)}(\pm 1) = 0,\quad \varphi_{2y}^{(0)}(\pm 1) = 0. \tag{21}$$

Similarly, collecting the terms proportional to AA^* we get

$$-\mathrm{Re}_c^{-1}\varphi_{2yyyy}^{(1)} = i\alpha_c(\varphi_{1y}\varphi_{1yy}^* - \varphi_{1y}^*\varphi_{1yy} + \varphi_1\varphi_{1yyy}^* - \varphi_1^*\varphi_{1yyy}) - \mathrm{Re}_c^{-1}U_{yyy} \tag{22}$$

with the boundary conditions

$$\varphi_2^{(1)}(\pm 1) = 0,\quad \varphi_{2y}^{(1)}(\pm 1) = 0. \tag{23}$$

In a similar manner, we obtain the equation for the function $\varphi_2^{(2)}$:

$$-\mathrm{Re}_c^{-1}[\varphi_{2yyyy}^{(2)} - 2\alpha_c^2\varphi_{2yy}^{(2)} + \alpha_c^4\varphi_2^{(2)} + i\alpha_c(U - c_c)(\varphi_{2yy}^{(2)} - \alpha_c^2\varphi_2^{(2)}) - i\alpha_c U_{yy}\varphi_2^{(2)} = c_g(\varphi_{1yy} - \alpha_c\varphi_1) - 2\alpha_c^2 c_c\varphi_1 + 3\alpha_c^2 U\varphi_1 - U\varphi_{1yy} + U_{yy}\varphi_1 + 4i\alpha_c\,\mathrm{Re}_c^{-1}(\varphi_{1yy} - \alpha_c^2\varphi_1) \tag{24}$$

with the boundary conditions

$$\varphi_2^{(2)}(\pm 1) = 0,\quad \varphi_{2y}^{(2)}(\pm 1) = 0. \tag{25}$$

Comparing (11) – (12) and (24) – (25) we see that the solution to (24), (25), namely, the function $\varphi_2^{(2)}$, is resonantly forced since the homogeneous equation which corresponds to (24) is satisfied at $\mathrm{Re} = \mathrm{Re}_c$, $\alpha = \alpha_c$ and $c = c_c$. Thus, (24) – (25) has a solution if and only if the right-hand side of (24) is orthogonal to all the eigenfunctions of the corresponding adjoint problem. The adjoint operator, L^a, and the adjoint eigenfunction, φ_1^a, are defined as follows:

$$\int_{-1}^{1}\varphi_1^a L(\varphi_1)dy = \int_{-1}^{1}\varphi_1 L^a(\varphi_1^a)dy = 0. \tag{26}$$

The adjoint eigenfunction is the solution of the equation

$$\varphi_{1yyyy}^a - (2\alpha^2 + i\alpha U\,\mathrm{Re})\varphi_{1yy}^a - 2i\alpha U_y\,\mathrm{Re}\,\varphi_{1y}^a + (i\alpha^3 U\,\mathrm{Re} + \alpha^4)\varphi_1 = -i\alpha\,\mathrm{Re}\,c(\varphi_{1yy}^a - \alpha^2\varphi_1^a) \tag{27}$$

with the boundary conditions

$$\varphi_1^a(\pm 1) = 0,\quad \varphi_{1y}^a(\pm 1) = 0. \tag{28}$$

Note that the critical values Re_c, α_c and c_c are the same for problems (11) – (12) and (27) – (28).

The group velocity, c_g, is determined from the solvability condition for equation (23) and is given by

$$c_g = \frac{I_1}{I_2}, \tag{29}$$

where

$$I_1 = \int_{-1}^{1}\varphi_1^a\begin{bmatrix} 2\alpha_c^2 c_c\varphi_1 - 3\alpha_c^2 U\varphi_1 + U\varphi_{1yy} - \varphi_1 U_{yy} \\ -4i\alpha_c\,\mathrm{Re}_c^{-1}(\varphi_{1yy} - \alpha_c^2\varphi_1)\end{bmatrix}dy$$

and

$$I_2 = \int_{-1}^{1} \varphi_1^a (\varphi_{1yy} - \alpha_c^2 \varphi_1) dy.$$

Finally, the amplitude evolution equation for A is obtained from the solvability condition for equation (18) and has the form of the complex Ginzburg-Landau equation (4) where

$$\sigma = \frac{\sigma_1}{\gamma_1}, \quad \delta = \frac{\delta_1}{\gamma_1}, \quad \mu = \frac{\mu_1}{\gamma_1}. \tag{30}$$

The coefficients $\gamma_1, \sigma_1, \delta_1$ and μ_1 are given by

$$\gamma_1 = \int_{-1}^{1} \varphi_1^a (\varphi_{1yy} - \alpha_c^2 \varphi_1) dy, \tag{31}$$

$$\sigma_1 = -\mathrm{Re}_c^{-1} \int_{-1}^{1} \varphi_1^a (\varphi_{1yyyy} - 2\alpha_c^2 \varphi_{1yy} + \alpha_c^4 \varphi_1) dy, \tag{32}$$

$$\delta_1 = \int_{-1}^{1} \varphi_1^a \begin{bmatrix} (c_g - U + 4i\alpha_c \mathrm{Re}_c^{-1})\varphi_{2yy}^{(2)} + (-\alpha_c^2 c_g \\ -2\alpha_c^2 c_c + 3\alpha_c^2 U + U_{yy} \\ -4i\alpha_c \mathrm{Re}_c)\varphi_2^{(2)} + 2\mathrm{Re}_c^{-1} \varphi_{1yy} \\ +(2i\alpha_c c_g + i\alpha_c c_c \\ -3i\alpha_c U - 6\alpha_c^2 \mathrm{Re}_c^{-1})\varphi_1 \end{bmatrix} dy, \tag{33}$$

$$\mu_1 = -i\alpha_c \int_{-1}^{1} \varphi_1^a \begin{bmatrix} -\varphi_1^* \varphi_{2yyy}^{(0)} - 2\varphi_{1y}^* \varphi_{2yy}^{(0)} \\ +(3\alpha_c^2 \varphi_1^* + \varphi_{1yy}^*)\varphi_{2y}^{(0)} \\ +(-10\alpha_c^2 \varphi_{1y}^* + 2\varphi_{1yyy}^*)\varphi_2^{(0)} \\ +\varphi_1 \varphi_{2yyy}^{(1)} - (\varphi_{1yy} - \alpha_c^2 \varphi_1)\varphi_{2y}^{(1)} \end{bmatrix} dy. \tag{34}$$

4 Numerical example

The coefficients of the Ginzburg-Landau equation are evaluated numerically in this section. First, the linear stability problem (11) – (12) is solved by means of a pseudospectral collocation method based on Chebyshev polynomials. The solution to (11) – (12) is sought in the form

$$\varphi(y) = \sum_{k=0}^{N} a_k (1 - y^2)^2 T_k(y), \tag{35}$$

where $T_k(y)$ is the Chebyshev polynomial of degree k. The form of (35) guarantees that the boundary conditions (12) are satisfied automatically. The collocation points y_j are

$$y_j = \cos\frac{\pi j}{N}, \quad j = 0,1,..., N. \tag{36}$$

High precision is necessary to calculate the coefficients of the Ginzburg-Landau equation. Therefore, the number of collocation points, N, is fixed at $N = 100$ in the present study. Numerical results indicate that such value of N is sufficient to calculate the coefficients to four decimal places. In order to illustrate the procedure, we have chosen the base velocity profile which corresponds to the dimensionless time $t = 0.0001$. The corresponding critical values of the parameters of the problem are $\mathrm{Re}_c = 2853.024$, $\alpha_c = 1.1527$ and $c_c = 0.35903$ (note that c_c is the real part of c and the imaginary part of c is of order 10^{-8} in this case).

Second, we calculate the eigenfunctions of the linear stability problem (11) – (12) and the eigenfunction of the corresponding adjoint problem (27) – (28). The eigenvalues of (27) – (28) are the same as the eigenvalues of (11) – (12), as it should be.

Third, the boundary value problems (20) – (21), (22) – (23) and (24) – (25) are solved by means of the Chebyshev collocation method and the functions $\varphi_2^{(0)}(y), \varphi_2^{(1)}(y)$ and $\varphi_2^{(2)}(y)$ are used to calculate the coefficients of the Ginzburg-Landau equation (4). The group velocity c_g calculated by (29) must be real. The calculations confirm this fact: the computed value of c_g is $c_g = -0.229730 - 0.000011i$. The coefficients σ, δ and μ of the Ginzburg-Landau equation (4) are

$$\sigma = 0.0244 + 0.0594i,$$
$$\delta = 0.1805 + 0.2835i,$$
$$\mu = 33.0005 - 253.5093i.$$

Since the real part of μ (the Landau constant) is positive, a finite equilibrium state is not possible. This means that the disturbances are linearly unstable and grow unbounded; that is, the stability is subcritical. A similar result is obtained in [13] for a weakly nonlinear analysis of plane Poiseuille flow.

5 Conclusion

Weakly nonlinear theory is used in the present paper to derive an amplitude evolution equation for the most unstable mode for suddenly blocked laminar flow in a plane channel between two parallel infinite plates. The quasi-steady assumption is used for linear stability analysis, that is, the growth rate of perturbations is assumed to be considerably larger than the rate of change of the base flow with respect to time. Amplitude evolution equations are considered as an effective tool for the analysis of spatio-temporal characteristics of complex flows. It is shown in the paper that the evolution equation in this case is the complex Ginzburg-Landau equation. The coefficients of the equation are calculated in closed form. Results of numerical calculations indicate that

unstable disturbances (computed at $t = 0.0001$) grow without bounds and a finite amplitude state is not possible.

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