Proceedings of the 3rd IASME/WSEAS Int. Conf. on HEAT TRANSFER, THERMAL ENGINEERING AND ENVIRONMENT, Corfu, Greece, August 20-22, 2005 (pp397-402)

# Hyperbolic heat conduction in a band irradiated by different types of laser sources

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*Abstract:*- In this work we compute the hyperbolic temperature profile produced in an infinite band which is irradiated by a laser source. We study the temperature response in different cases according to the type of the laser applied: The temporal profile can be continuous or a rectangular pulse, and the spatial profile can be Gaussian or doughnut-shaped. The temperature profiles in each case are computed from Green's function of the Neumann problem for the axially symmetric hyperbolic heat conduction equation in an infinite band.

*Key-words:* - Hyperbolic heat equation, irradiated band, laser.

## 1 Introduction

We are interested in the temperature response in an infinite band which is irradiated by different types of laser sources. Due to the physical environment of the problem, the more interesting solutions are those ones being axially symmetric with respect to the vertical $z$-axis. Actually, it is well known that the accurate study of these processes needs the use of the hyperbolic heat conduction equation, more involved than the classical parabolic Fourier equation.

The formulation of this kind of problems usually implies the study of a Neumann problem for the hyperbolic heat equation in which the boundary or initial conditions are given by irregular distributions such as the Heaviside or the Dirac $\delta$ functions. However, in spite of being very "complex" problems, in the most of the cases the temperature computations are only made in a purely formal way without doing any theoretic reasoning.

In [7] we have developed a rigorous mathematical study of the hyperbolic heat conduction equation in order to provide the theoretical foundations for temperature computation in different heat conduction problems, including those whose initial and boundary conditions are given by irregular distributions. The basis of this mathematical treatment is the study of the Green's function of the Neumann problem for the hyperbolic heat equation. At the end of this study we have computed the Green's function of the Neumann problem for the axially symmetric hyperbolic heat equation in an infinite band and we have found an expression that relates the temperature solution of a heat conduction problem in the band with its Green's function.

The objective of this work is to use the results of [7] related to Green's function in order to obtain the temperature response in an infinite band irradiated by different types of laser sources: The temporal profile can be continuous or a rectangular pulse, and the spatial profile can be Gaussian or doughnut-shaped. This problem has been solved in [3] in a different way. At the end of this work we discuss the solution found in [3] and we compare it with our solution.

## 2 Analytical development

The hyperbolic heat conduction problem is based in the irradiation of an infinite band by a laser

[1]The research of the authors was partially supported by the MEC and FEDER projects MTM2004-02262 and MTM2004-21420-E, and AVCIT group 03/050.

source. In cylindrical coordinates the governing equation of the problem is

$$-\frac{\partial^2 T}{\partial r^2} - \frac{1}{r}\frac{\partial T}{\partial r} - \frac{\partial^2 T}{\partial z^2} + \frac{1}{\alpha}\frac{\partial T}{\partial t} + \frac{\tau}{\alpha}\frac{\partial^2 T}{\partial t^2} = 0$$

for $(\rho, \eta, \xi) \in ]0, \infty[\times]0, L_0[\times]0, \infty[$, where $T$ is the temperature and $\alpha$ and $\tau$ are the thermal diffusivity and the thermal relaxation time of the medium, respectively. The initial and boundary conditions of the problem are

$$\begin{aligned}
&\lim_{t\to\infty} T(r,z,t) = \lim_{t\to\infty} \frac{\partial T}{\partial t}(r,z,t) = 0 \,, \\
&T(r,z,0) = \frac{\partial T}{\partial t}\,(r,z,0) = 0, \quad \text{in} \quad ]0,\infty[\times]0,L[\,, \\
&\frac{\partial\, T}{\partial\, z}(r,0,t) = \frac{Q_0}{k}\left(f(t) + \tau\,\frac{\partial\, f(t)}{\partial\, t}\right) \\
&\qquad\qquad \times \left(A_0 + (1-A_0)\frac{r^2}{d^2}\right) e^{-\frac{r^2}{d^2}} \,, \qquad (1) \\
&\frac{\partial T}{\partial z}(r,L,t) = 0 \quad \text{in} \quad ]0,\infty[\times]0,\infty[\,, \\
&\frac{\partial T}{\partial r}(0,z,t) = 0 \quad \text{in} \quad ]0,L[\times]0,\infty[\,,
\end{aligned}$$

where $k$ is the conductivity, $Q_0$ is a factor corresponding to the maximum incident flux for a Gaussian source and is related to surface physics, $d$ is a characteristic beam radius which represents the circular boundary within the Gaussian source that contains 63% of the total beam power incident to the surface, $f(t)$ is the function that determines the laser temporal profile and $A_0$ is the fraction of the total flux that contains the Gaussian mode and varies from 0 to 1. When the laser spatial profile is Gaussian the maximum irradiation is found at the center and $A_0 = 1$, this profile is employed in material processing applications involving high reflective metallic surfaces. When the laser spatial profile is doughnut the maximum irradiation is found in a ring around the center and $A_0 = 0$, this profile is employed in some processes of materials cut.

For convenience we work with following dimensionless variables

$$\begin{aligned}
&\rho := \frac{r}{2\sqrt{\alpha\tau}} \;;\; \eta := \frac{z}{2\sqrt{\alpha\tau}} \;;\; \xi := \frac{t}{2\tau} \;;\; \mu := \frac{2\sqrt{\alpha\tau}}{d} \\
&V(\rho,\eta,\xi) := \frac{T(r,z,t)\,k}{Q_0\sqrt{\alpha\tau}} \;;\quad L_0 := \frac{L}{2\,\sqrt{\alpha\,\tau}} \,. \qquad (2)
\end{aligned}$$

Bearing in mind these variables the problem that we want to solve is

$$-\frac{\partial^2 V}{\partial \rho^2} - \frac{1}{\rho}\frac{\partial V}{\partial \rho} - \frac{\partial^2 V}{\partial \eta^2} + 2\,\frac{\partial V}{\partial \xi} + \frac{\partial^2 V}{\partial \xi^2} = 0$$

for $(\rho, \eta, \xi) \in ]0, \infty[\times]0, L_0[\times]0, \infty[$, verifying the following initial and boundary conditions

$$\begin{aligned}
&\lim_{\xi\to\infty} V(\rho,\eta,\xi) = \lim_{\xi\to\infty} \frac{\partial V}{\partial\,\xi}(\rho,\eta,\xi) = 0 \\
&V(\rho,\eta,0) = \frac{\partial V}{\partial \xi}\,(\rho,\eta,0) = 0 \quad \text{in} \quad ]0,\infty[\times]0,L_0[\,, \\
&\frac{\partial V}{\partial \eta}(\rho,0,\xi) = 2F(\xi)\left(A_0 + (1-A_0)\mu^2\rho^2\right) e^{-\rho^2\mu^2} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3) \\
&\frac{\partial V}{\partial \eta}(\rho,L_0,\xi) = 0 \quad \text{in} \quad ]0,\infty[\times]0,\infty[\,, \\
&\frac{\partial V}{\partial \rho}\,(0,\eta,\xi) = 0 \quad \text{in} \quad ]0,L_0[\times]0,\infty[\,.
\end{aligned}$$

Given these initial and boundary conditions, according to [4] the formula which relates the temperature and Green's function for this problem is

$$\begin{aligned}
V(\rho,\eta,\xi) = &\int_0^\infty \int_0^\infty \rho_0\, \Theta(\rho,\eta,\xi|\rho_0,0,\xi_0) \\
&\times \frac{\partial V}{\partial \eta}(\rho,0,\xi)\; d\rho_0\; d\xi_0, \qquad (4)
\end{aligned}$$

where $\Theta(\rho, \eta, \xi|\rho_0, \eta_0, \xi_0)$ is Green's function corresponding to the Neumann problem for the axially symmetric hyperbolic heat equation in an infinite band. In [4] we have obtained this Green's function which is

$$\begin{aligned}
&\Theta(\rho,\eta,\xi|\rho_0,\eta_0,\xi_0) = \frac{1}{L_0} H(\xi-\xi_0) e^{-(\xi-\xi_0)} \\
&\times \left[ \int_0^1 g(k,\rho,\rho_0) \frac{\sinh\left((\xi-\xi_0)\sqrt{1-k^2}\right)}{\sqrt{1-k^2}} dk \right. \\
&+ \int_1^\infty g(k,\rho,\rho_0) \frac{\sin\left((\xi-\xi_0)\sqrt{k^2-1}\right)}{\sqrt{k^2-1}} dk \\
&+ \sum_{n=1}^\infty \cos\frac{n\pi\eta}{L_0} \cos\frac{n\pi\eta_0}{L_0}\,(H(D(n)) \\
&\times \left[ \int_0^{\sqrt{D(n)}} 2g(k,\rho,\rho_0) \frac{\sinh\left((\xi-\xi_0)\sqrt{B(k,n)}\right)}{\sqrt{B(k,n)}} dk \right.
\end{aligned}$$

$$+ \int_{\sqrt{D(n)}}^{\infty} 2g(k,\rho,\rho_0) \frac{\sin\left((\xi-\xi_0)\sqrt{-B(k,n)}\right)}{\sqrt{-B(k,n)}} dk\Bigg]$$

$$+ H(-D(n)) \int_0^{\infty} 2g(k,\rho,\rho_0)$$

$$\times \frac{\sin\left((\xi-\xi_0)\sqrt{-B(k,n)}\right)}{\sqrt{-B(k,n)}} dk \Bigg)\Bigg], \quad (5)$$

where $g(k,\rho,\rho_0) := kJ_0(k\rho)J_0(k\rho_0)$, $D(n) := 1 - \frac{n^2\pi^2}{L_0^2}$ and $B(k,n) := D(n) - k^2$.

# 3 Analysis

We solve the problem assuming four different cases depending on the laser spatial and temporal profile.

## 3.1 Continuous doughnut laser source

If the laser is continuous and doughnut $f(t) = H(t)$ and $A_0 = 0$, where $H(t)$ is the Heaviside function. Then, the dimensionless form of condition (1) is

$$\frac{\partial V}{\partial \eta}(\rho, 0, \xi) = (2\, H(\xi) + \delta(\xi))\ \mu^2\rho^2\, e^{-\mu^2\rho^2}. \quad (6)$$

In order to compute the temperature profile in this case we put (6) and (5) into formula (4). Moreover, to simplify the resulting expression we use formula 9.210 in [2]. In this way we obtain the temperature profile:

$$V(\rho,\eta,\xi) = \frac{1}{L_0\mu^2}\left[\int_0^{\xi} e^{-(\xi-\xi_0)} F_1(\rho,\eta,\xi-\xi_0)\, d\xi_0 + \frac{e^{-\xi}}{2} F_1(\rho,\eta,\xi)\right] \quad (7)$$

where

$$F_1(\rho,\eta,\xi) = \sum_{n=1}^{\infty} \cos\frac{n\pi\eta}{L_0}\, (H(-D(n))$$

$$\times \int_0^{\infty} 2kJ_0(k\rho) e^{L(k)-1} L(k) \frac{\sin\left(\xi\sqrt{-B(k,n)}\right)}{\sqrt{-B(k,n)}} dk$$

$$+ H(D(n)) \Bigg( \int_0^{\sqrt{D(n)}} 2kJ_0(k\rho) e^{L(k)-1} L(k)$$

$$\times \frac{\sinh\left(\xi\sqrt{B(k,n)}\right)}{\sqrt{B(k,n)}} dk + \int_{\sqrt{D(n)}}^{\infty} 2kJ_0(k\rho)$$

$$\times e^{L(k)-1} L(k) \frac{\sin\left(\xi\sqrt{-B(k,n)}\right)}{\sqrt{-B(k,n)}} dk \Bigg)\Bigg)$$

$$+ \int_0^1 kJ_0(k\rho) \frac{\sinh\left(\xi\sqrt{1-k^2}\right)}{\sqrt{1-k^2}} e^{L(k)-1} L(k)\, dk$$

$$+ \int_1^{\infty} kJ_0(k\rho) \frac{\sin\left(\xi\sqrt{k^2-1}\right)}{\sqrt{k^2-1}} e^{L(k)-1} L(k) dk$$

where $L(k) = 1 - \frac{k^2}{4\mu^2}$.

## 3.2 Continuous Gaussian laser source

If the laser is continuous and Gaussian $f(t) = H(t)$ and $A_0 = 1$. Then, the dimensionless form of condition (1) is

$$\frac{\partial V}{\partial \eta}(\rho, 0, \xi) = (2\, H(\xi) + \delta(\xi))\ e^{-\mu^2\rho^2}. \quad (8)$$

As in the previous case, to obtain the temperature profile we put (8) and (5) into formula (4) and use formula 4 of section 6.631 in [2]. Following these steps we get the desired temperature profile:

$$V(\rho,\eta,\xi) = \frac{1}{L_0\mu^2}\left[\int_0^{\xi} e^{-(\xi-\xi_0)} F_2(\rho,\eta,\xi-\xi_0)\, d\xi_0 + \frac{e^{-\xi}}{2} F_2(\rho,\eta,\xi)\right] \quad (9)$$

where

$$F_2(\rho,\eta,\xi) = \int_0^1 kJ_0(k\rho) \frac{\sinh\left(\xi\sqrt{1-k^2}\right)}{\sqrt{1-k^2}} e^{L(k)-1}\, dk$$

$$+ \int_1^{\infty} kJ_0(k\rho) \frac{\sin\left(\xi\sqrt{k^2-1}\right)}{\sqrt{k^2-1}} e^{L(k)+1}\, dk$$

$$+ \sum_{n=1}^{\infty} \cos\frac{n\pi\eta}{L_0} \Bigg( H(-D(n)) \int_0^{\infty} 2kJ_0(k\rho) e^{L(k)-1}$$

$$\times \frac{\sin\left(\xi\sqrt{-B(k,n)}\right)}{\sqrt{-B(k,n)}}\, dk + H(D(n))$$

$$\times\left(\int_0^{\sqrt{D(n)}} 2kJ_0(k\rho)e^{L(k)-1}\frac{\sinh\left(\xi\sqrt{B(k,n)}\right)}{\sqrt{B(k,n)}}\,dk\right.$$

$$\left.+\int_{\sqrt{D(n)}}^{\infty} 2kJ_0(k\rho)e^{L(k)-1}\frac{\sin\left(\xi\sqrt{-B(k,n)}\right)}{\sqrt{-B(k,n)}}dk\right)\Bigg).$$

### 3.3 Single pulse doughnut laser source

Since the spatial profile is doughnut-shaped as in section 3.1 $A_0 = 0$. And the temporal profile is $f(t) = H(t) - H(t - \Delta t)$, where $\Delta t$ represents the application time of the laser pulse. Introducing these values into (1) and taking dimensionless variables we get the following condition

$$\frac{\partial V}{\partial \eta}(\rho, 0, \xi) = [2\,(H(\xi) - H(\xi - \Delta\xi)) \\ +\delta(\xi) - \delta(\xi - \Delta\xi)]\,\mu^2\rho^2 e^{-\mu^2\rho^2}. \quad (10)$$

To solve the problem we put condition (10) into (4). Then, we follow the same steps than in section 3.2. We have to be careful in this case with the simplification of Dirac delta and Heaviside functions. Finally, we get the temperature response

$$V(\rho,\eta,\xi) = \frac{1}{L_0\mu^2}\left[\int_0^{\xi} e^{-(\xi-\xi_0)}F_1(\rho,\eta,\xi-\xi_0)\,d\xi_0 \right.$$
$$+\frac{e^{-\xi}}{2}F_1(\rho,\eta,\xi) - H(\xi-\Delta\xi)\frac{e^{-\xi+\Delta\xi}}{2}F_1(\rho,\eta,\Delta\xi)$$
$$\left. -H(\xi-\Delta\xi)\int_{\Delta\xi}^{\xi} e^{-(\xi-\xi_0)}F_1(\rho,\eta,\xi-\xi_0)d\xi_0\right]. \quad (11)$$

### 3.4 Single pulse Gaussian laser source

Being a Gaussian spatial profile $A_0 = 1$ and the temporal profile is represented as in the previous case by $f(t) = H(t) - H(t - \Delta t)$. Putting these values into (1) and using dimensionless variables we obtain in this case

$$\frac{\partial V}{\partial \eta}(\rho, 0, \xi) = [2\,(H(\xi) - H(\xi - \Delta\xi)) \\ +\delta(\xi) - \delta(\xi - \Delta\xi)]\,e^{-\mu^2\rho^2}. \quad (12)$$

The solution of this problem is obtained following the same steps than in section 3.2. Now, we have to bear in mind that we have to put equation (12) into (4) instead of (8) and we have to be careful simplifying Dirac delta and Heaviside functions. In this way we get the desired temperature profile

$$V(\rho,\eta,\xi) = \frac{1}{L_0\mu^2}\left[\int_0^{\xi} e^{-(\xi-\xi_0)}F_2(\rho,\eta,\xi-\xi_0)\,d\xi_0 \right.$$
$$+\frac{e^{-\xi}}{2}F_2(\rho,\eta,\xi) - H(\xi-\Delta\xi)\frac{e^{-\xi+\Delta\xi}}{2}F_2(\rho,\eta,\Delta\xi)$$
$$\left. -H(\xi-\Delta\xi)\int_{\Delta\xi}^{\xi} e^{-(\xi-\xi_0)}F_2(\rho,\eta,\xi-\xi_0)d\xi_0\right]. \quad (13)$$

## 4 Graphic discussion

We use the software *Mathematica* to discuss the obtained temperature profiles. We assume that the band is made with a refractory material whose physical properties are $\alpha = 8.1\ 10^{-6}\ m^2/s$ and $\tau = 2.9\ 10^{-11}\ s$ (see [6]). Moreover, we assume $d = 2.45\ 10^{-6}\ m$ (see [1]) and $L = 3\ 10^{-8}\ m$. The value of the variables are those in which hyperbolic model has sense: $t$ between nano or picoseconds and $r$ and $z$ between nano or micrometers.

Firstly, figure 1 shows the temperature distribution as a function of $\rho$ in the case of a continuous laser source on the surface $\eta = 0$ at different times. We can see that the temperature has a doughnut behavior since the maximum temperature is reached in a ring around the center of the band (not in the center). Moreover, if the time increases, the temperature of a point increases too.

Figure 2 shows the temperature variation as a function of $\rho$ in the case of a continuous Gaussian laser source on the surface $\eta = 0$ at different times. The temperature distribution follows a Gaussian profile since the maximum temperature is reached at the center of the band. We can also observe that greater is the time, higher is the temperature reached.

Figure 3 depicts the temperature distribution in the center of the band as a function of $\eta$ at different times in the case of a continuous Gaussian laser source. We can see that for $\xi = 0.6$ there exists a spatial interval in which the temperature

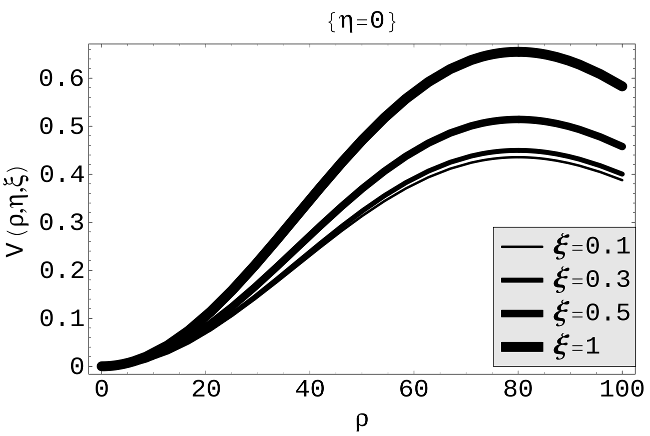


Figure 1: Temperature distribution as a function of $\rho$ from a continuous doughnut laser at $\eta = 0$ for different times.

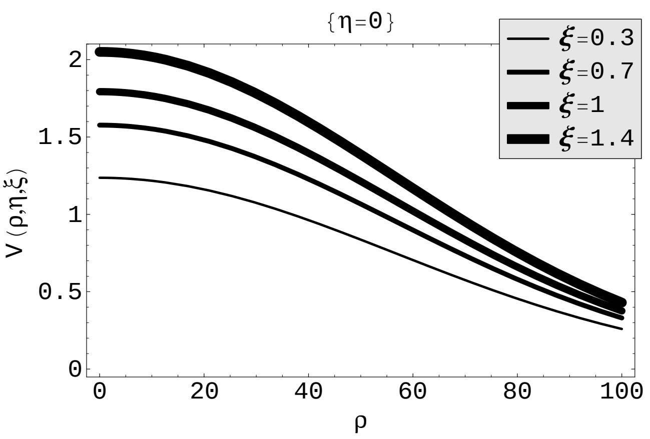


Figure 2: Temperature distribution as a function of $\rho$ from a continuous Gaussian laser at $\eta = 0$ for different times.

is zero since the heat flux has not arrived, and another zone where the temperature is bigger than zero since the heat flux has arrived. This fact is due to the consideration of a finite speed of heat conduction. For $\xi = 1.2$ we can observe the thermal wave has reached the insulated boundary at $\eta = 1$, and reflected. This effect is showed by the jump in temperature at $\eta = 0.8$.

Figure 4 shows the temperature distribution in the center of the band as a function of $\xi$ in the case of a pulse Gaussian laser source on the surface $\eta = 1$. Firstly in figure 4 we observe the existence of a zone where the temperature is zero since at this time the heat flux has not arrived at $\eta = 1$ and suddenly (at $\xi = 1$) the temperature increases because the heat flux produced by the application of

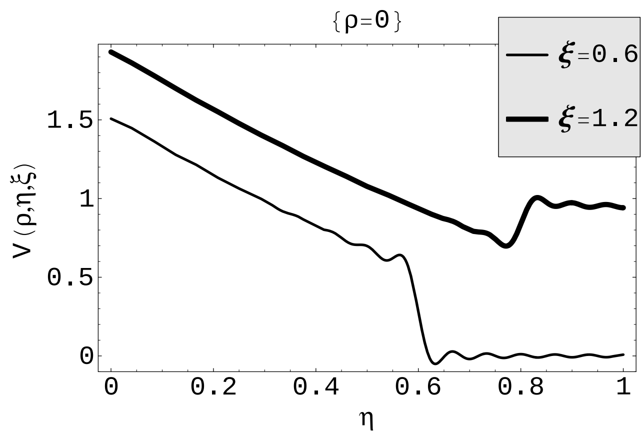


Figure 3: Variation of center temperature as a function of $\eta$ from a continuous Gaussian laser at different times

the laser has reached this point. We have assumed that the application time of the laser is $\Delta\xi = 0.2$, then during this time the temperature is increasing and after this interval decreases. Again, initially all the points of the band are not a higher temperature than the initial due to the consideration of a finite speed of heat conduction. We can notice at $\xi = 3$ that temperature increases again, this is due to the wave which has reached the insulated boundary and it is reflected.

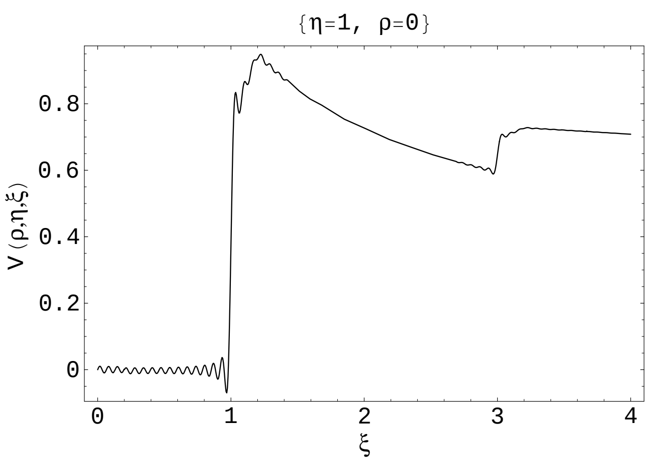


Figure 4: Variation of center temperature as a function of $\xi$ from a pulse Gaussian source on the surface $\eta = 1$.

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## 5 Comparison with the solution found in [3]

As we have said at the introduction this problem has been solved in [3] in a different way. Apart from the fact that we have based our computation in rigorous theoretical foundations, we have observed other differences between our solution and the solution found in [3].

One of the main differences is that the solution of the problem in every case that we provide is valid for large and small values of time whereas the solution of [3] is obtained making a simplification and it is only valid for small times.

Regarding to the computations, our solution it is obtained easier and more quickly. In our case it is not necessary to solve all the problem in every case, we only have to bear in mind the flux condition in each case and put it into the formula which relates temperature and Green's function.

According to the graphics we have also observed differences when we want to make a graphic with the temperature profiles found in [3] (for example in the case of a continuous Gaussian laser source). In this case we have to consider a large number of terms in the summations whereas in our case the convergence of the summation is faster what allow us to obtain the figures in less time.

## 6 Conclusions

In this paper we have obtained the hyperbolic solution of a heat conduction problem based on the irradiation of a an infinite band by different types of laser sources. The solution is found from a previous research that we have made about Green's function and the expression that relates this function and the temperature for a specific problem. Based in this previous study the solutions in such case are easily obtained and they are valid for all times.

As we can see with the graphics, the temporal profiles that we get show some of the main characteristics of the hyperbolic model: a finite speed of heat conduction and a its wave nature.

We compare our solution with the obtained in [3] by a different way. Our solution is easier to obtain and represent than [3], and moreover, it is based on theoretical foundations and valid for all values of time and not only for small ones as in [3].

## References

[1] BACHS, L., CUESTA, J., CARLES, N.: *Aplicaciones industriales del láser.* Macombo. Barcelona, (1988).

[2] GRADSHTEYN, I. S., RYZHIK, I. M.: *Table of integrals, series and products.* Academic Press. New York, (1980).

[3] HECTOR JR., L. G., KIM, W. S., ÖZISIK, M. N.: *Propagation and reflection of thermal waves in a finite medium due to axisymmetric surface sources.* Int. J. Heat Mass Transfer, pp. 897-912, 1992.

[4] LOPEZ MOLINA, J. A., TRUJILLO, M.: *Green's function of the Neumann problem for the anisotropic hyperbolic heat equation in a three-dimensional bounded domain,* Preprint, (2004).

[5] LOPEZ MOLINA, J. A., TRUJILLO, M.: *Green's function of the Neumann problem for the hyperbolic heat equation in a three-dimensional band,* Preprint, (2004).

[6] MAURER, M. J., THOMPSON, H. A.: *Non Fourier effects at high heat flux.* Journal of heat transfer. Trans. ASME, 284-286, (1973).

[7] TRUJILLO, M. *Función de Green para la ecuación hiperbólica de transmisión del calor.* Doctoral dissertation, (2005).