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Characterization of 1-D and 2-D Cylindrical Pin Fins with Adiabatic and Convective End from the Characteristic Length

[1]LUNA ABAD, J.P. and [2]ALHAMA, F.
[1]Department of Thermal Engineering and Fluids, [2]Department of Applied Physics
Polytechnic University of Cartagena
ETSII, Campus Muralla del Mar, Cartagena
SPAIN
Paco.alhama@upct.es http://www.upct.es

Abstract: - Although the cylindrical pin fin has been widely treated in thermal engineering literature, the problem has not been approached from the (two) dimensionless numbers provided by the discriminate dimensional analysis, particularly the characteristic length. The meaning of this dimensionless parameter is clear. Characterization of these fins based on the characteristic length is carried out for 1-D and 2-D conduction hypothesis and adiabatic and convective ends. Analytical and numerical solutions are used for this characterization. A Biot number that approximates the solutions of convective and adiabatic end is determined. Efficiency graphs are also presented.

Key-Words: - Discriminated dimensional analysis, cylindrical pin fin, characteristic length, efficiency, 1-D and 2-D conduction, transversal Biot number.

1 Introduction

Much equipment, such as evaporator coils in air-conditioners or simple heat sinks in electronic devices, make use of pin fins of cylindrical geometry. A large body of literature describes in detail heat transfer through these fins and even the optimization design [1]. Generally, cylindrical pin fins are studied under the hypothesis of 1-D conduction or, what it is assumed as the same condition, for small values of transversal Biot numbers. However, while 1-D analytical solution depends on hyperbolic functions, 2-D solutions, which are much more complex, are expressed in terms of an infinite series of Bessel functions [2]. Irey studied the errors derived assuming a 1-D hypothesis for a wide range of both the transverse Biot numbers and the aspect ratio of the fin, a range that goes beyond any practical applications.

In this work, a characterization of cylindrical pin fins with adiabatic or convective end is established starting from the (two) dimensionless parameters provided by the application of discriminated dimensional analysis [3] to these fins, i.e., the transversal Biot number, $Bi_t = hR/2k$, and the ratio l/l^*, l the real length of the fin and $l^*=(kr/2h)^{1/2}$ the "characteristic length". As we shall demonstrate l^*, a "hidden" quantity obtained by discriminate dimensional analysis, has a clear significance.

Firstly the thermal significance of l^* is presented based on the temperature profile and/or the heat dissipated from the fin per unit of length. After that, the fin efficiency is theoretical and numerically evaluated in 1-D pin fins with adiabatic or convective end for different values of Bi_t. Finally, a 2-D model based on the network simulation method is simulated to solve pin fins heat conduction problem under adiabatic or convective end and different Bi_t. A particular boundary for Bi_t that brings together the solutions for adiabatic and convective ends is determined.

2 Problem Formulation

2-D heat conduction in a cylindrical pin fin, fig.ure 1, is governed by the mathematical model:

$$(1/r)\partial(kr(\partial\theta/\partial r))/\partial r+\partial(k(\partial\theta/\partial z))/\partial z = \rho c_e(\partial\theta/\partial t)$$

$$0<z<L,\ 0<r<R \quad (1)$$

$$k(\partial\theta/\partial r) = h(\theta - \theta_{ref}),\ r=R,\ 0<z<L \quad (2)$$

$$\partial\theta/\partial r = 0,\ r=0,\ 0<z<L \quad (3)$$

$$\theta = \theta_b,\ 0<r<R,\ z=0 \quad (4)$$

$$\partial\theta/\partial r = 0,\ 0<r<R,\ z=L \quad (5a)$$

$$k(\partial\theta/\partial r) = h(\theta - \theta_{ref}),\ 0<r<R,\ z=L \quad (5b)$$

$$\theta = \theta_o,\ 0<z<L,\ 0<r<R,\ t=0 \quad (6)$$

Equation (1) is the heat conduction equation while equations (2-5) are the boundary conditions, (5a) represents the adiabatic end condition while (5b) is the convective end. Finally, equation (6) is the initial condition.

In these equations, θ is the temperature (K), k the thermal conductivity ($Wm^{-1}K^{-1}$), ρc_e the specific heat ($Jkg^{-1}K^{-1}$), h the heat transfer or convective coefficient and θ_b, θ_{ref}, θ_o the temperatures at the base, surrounding fluid far from the fin surface, and the initial temperature of the fin, respectively.

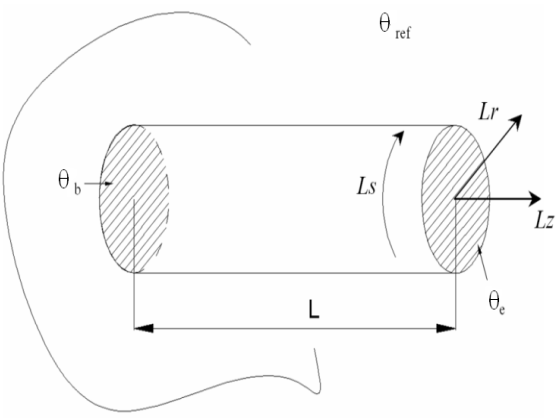

Fig. 1. Two dimensional cylindrical pin fin

3 The dimensionless numbers

The application of classical dimensional analysis to this problem such as it is carried out in most of textbooks [4,5], would lead to monomials like hL/k (a kind of Biot number). Even thou the monomial hL/k is dimensionless, the solution of the problem does not depend on it, i.e., they do not play an independent role in the solutions. Some questions arise from this subject: Why do these "dimensionless" monomials not play a fundamental part in the solution? Are they really dimensionless? What is wrong?

One of the keys to understanding this controversial subject is that classical dimensional analysis is a scalar theory. However, in most physical problems of interest, quantities and physical properties have a non-scalar character. Dimensional analysis theory should be a vectorial (discriminated) theory. That is, quantities such as lengths, forces, flows, etc, as well as specific properties of the material, such as thermal conductivity and diffusivity, would have different dimensional equations for each spatial direction. In this sense, the ratio "length of the fin/radius of the transversal surface", for example, would be a "discriminate" non-dimensionless number.

Apart from this question, the list of the relevant variables from which the dimensionless numbers derive has to be exhaustive, containing no more and no less than those justified by the problem. Regarding this point, the fin length might (or not) be introduced into the relevant list of variables, depending on (or not) the influence of this length in the results, a question that might not be known a priori. In this sense, what role does a very long pin fin length play in the solution of the temperature and heat flux fields? Probably, a very minor one.

After all, let us form the following list of relevant quantities and physical properties for the solution of the steady state problem (for the moment, we will introduce L into the list):

$$k_r,\ k_z,\ h_l,\ L,\ R \text{ and } \Delta\theta\ (\Delta\theta = \theta_b - \theta_{ref}) \qquad (7)$$

Since the conduction takes place in the radial and axial direction, thermal conductivity has different dimensional equations for each direction, even if its numerical value is the same (isotropy fin). The adequate dimensional basis for this problem is [3],

$$\{L_s, L_r, L_z, Q, \theta, T\}$$

where L_s, L_r, and L_z denote the dimension of the length according to the circumferential, radial or axial directions, respectively (Fig. 1) Q is the dimension of the heat, and θ and t denote the dimensions of temperature and time, respectively. Mass, M, is not included in the basis, since no inertial effects exist in the problem. Other dimensional basis could have been chosen in which quantities such as surfaces or angles, as well as lengths, could have been included [3]. As is known, the only requirement that a dimensional basis requires is that the set of dimensions is complete and that the quantities that they represent are independent of one another.

The dimensional equations of the variables of the relevant list are:

$$[k_r] = Q\theta^{-1}T^{-1}L_s^{-1}\ L_z^{-1}L_r$$
$$[k_z] = Q\theta^{-1}T^{-1}L_s^{-1}\ L_r^{-1}L_z$$
$$[h_l] = Q\theta^{-1}T^{-1}L_s^{-1}\ L_z^{-1}$$
$$[R] = L_r$$
$$[L] = L_z$$
$$[\Delta\theta] = \theta$$

From them, the two dimensionless monomials, immediately deduced by applying Buckhingam pi theorem, are

$\pi_1 = h_l R/k_r$

$\pi_2 = L/(k_z R/h_l)^{1/2}$

As is known, π_1 is the transversal Biot number, currently defined in cylindrical coordinates as $Bi_t = h_l R/2k$. On the other hand, the denominator of π_2 has the dimensions of a length in the axial direction, $l_z^* = (k_z R/h_l)^{1/2}$, which can be interpreted as a "characteristic axial length", in turn, independent of the actual value of the pin fin length, L.

In effect, if we consider a very long pin fin, the influence of L in the solution is quite negligible, so that L cannot take part in the solution. Eliminating L from the relevant list, the only pi monomial that can be formed is π_1 (or Bi_t); nevertheless, a hidden quantity [5], l^*, could be derived from the list, whose value is

$l_z^* = (k_z R/h_l)^{1/2}\ \varphi(Bi_t)$

where, as usual in the dimensional analysis solution, $\varphi(Bi_t)$ is an unknown function of Bi_t. The physical significance of l_z^*, as we shall see, is the order of magnitude of the pin fin length, in which a sensible part of the total heat transfer of the fin occurs. Since l_z^* does not depend on the boundary condition at the fin base, this length is the same both for an isothermal condition and for a constant incident heat flux condition at the base (whatever θ_b or j_b).

In conclusion, for a "long" isothermal-base pin fin the profile of the dimensionless temperature "$(\theta-\theta_{ref})/(\theta_b-\theta_{ref})$", the profile of the dimensionless heat flux "j/j_b" and the profile of the accumulated dimensionless heat flux "j_{ac}/j_b" as a function of the dimensionless coordinate "z/l^*" (considering these profiles separately) are identical.

When the pin fin is not "long", that is when L is of the order of magnitude of l^*, and the fin end is adiabatic, the relevant list is given by (7) and the solution depends on two dimensionless monomials $h_l R/k_r$ and $L/(k_z R/h_l)^{1/2}$; eventually if $h_l R/k_r$ is small the (1-D) solution depends only on the monomial $L/l^* = L/(k_z R/hl)^{1/2}$. On the other hand if the fin is not long and the fin end is convective, the relevant list is given by

k_r, k_z, h_l, h_e, R, L and $\Delta\theta$ ($\Delta\theta = \theta_b - \theta_{ref}$) (8)

with $[h_e] = Q\theta^{-1}T^{-1}L_s^{-1}\ L_r^{-1}$.

From this set of variables three dimensionless monomials can be formed, namely

$\pi_1 = h_l R/k_r$

$\pi_2 = L/(k_z R/h_l)^{1/2}$

$\pi_3 = h_l L/h_e R$

If numerically, $h_l = h_a$ and $k_r = k_z$, as is generally assumed, any one of these three monomials depends of the other two, for example $\pi_3^2 = \pi_2^2/\pi_1$. As a consequence of this connection between π_1, π_2 and π_3, even if the problem is quite different from an adiabatic end problem, the solutions will depend on the same two monomials.

4 The characteristic length, l*

The solutions of temperatures and heat flux fields in the fin have been obtained numerically from the network model of the problem [6,7]. We have proved that these solutions are quite identical to analytical solutions but easier to evaluate.

Fig. 2 represents the steady state dimensionless temperature profile of two very long cylindrical 2-D pin fins with adiabatic ends (a boundary condition that does not influence the solution) defined by the values (SI):

Pin fin 1:
$k_1 = 50$, $h_1 = 10$,
$L_1 = 0.5$, $R_1 = 0.01$,
$\theta_{b,1} = 20$ºC, $\theta_{ref,1} = 10$ºC,
$Bi_{t,1} = h_1 R_1/2k_1 = 0.0001$, $l^*_1 = (k_1 R_1/2h_l)^{1/2} = 0.05$,

Pin fin 2:
$k_2 = 100$, $h_2 = 30$,
$L_2 = 1$, $R_2 = 0.006$,
$\theta_{b,2} = 60$ºC, $\theta_{ref,2} = 10$ ºC--,
$Bi_{t,2} = h_2 R_2/2k_2 = 0.0009$, $l^*_2 = (k_2 R_2/2h_2)^{1/2} = 0.1$,

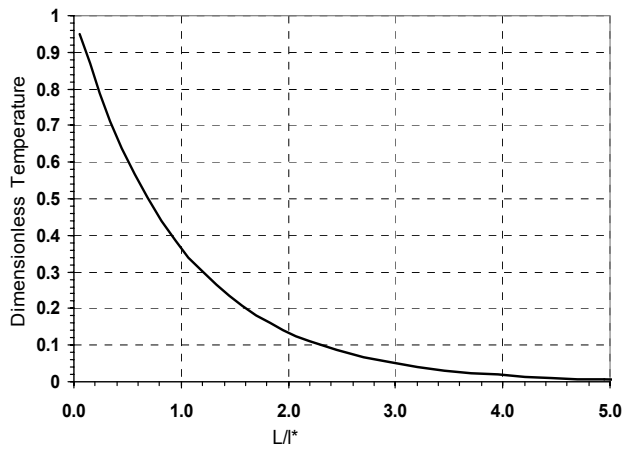

Fig. 2. Dimensionless temperature profiles

These values provides negligible transversal Biot numbers in order to eliminate the influence of Bi_t in the profile. Horizontal scale is the dimensionless coordinate z/l^*_1 or z/l^*_2. As may be seen, the profiles are quite identical and they are also identical for any value of r, for 0<r<R. These profiles suppose that these pin fins satisfy the 1-D conduction hypothesis.

Fig. 3 and 4 show the local and accumulated dimensionless (convective) heat flux, respectively. Again, the heat flux profile is identical for both fins.

From the figures, a universal definition of l^* arises. l^* is the length for which

i) the temperature reduces to 0.3864 times $\Delta\theta$, or

ii) the accumulated heat flux is 0.649 the total dissipated heat flux.

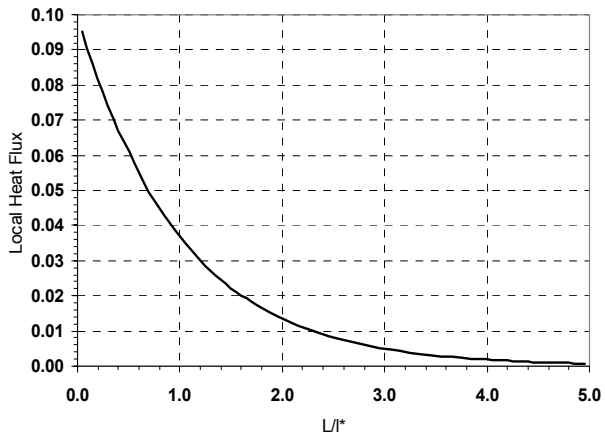

Fig. 3. Dimensionless local (convective) heat Flux profiles

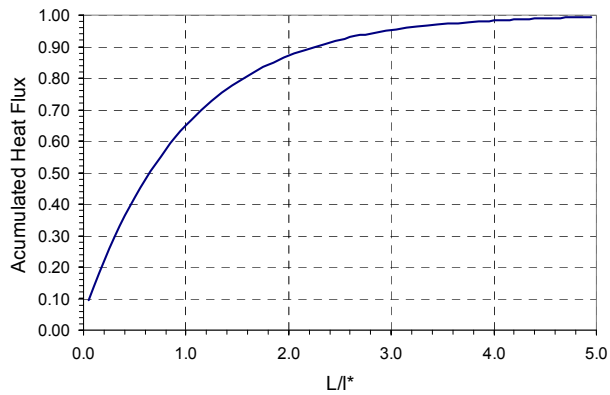

Fig. 4. Dimensionless (convective) accumulated heat flux profiles

	l^*	$2\,l^*$	$3\,l^*$	$5\,l^*$
$\Delta\theta$ reduction factor	0.3864	0.1422	0.0523	0.01926
j accumulated reduction factor	0.6495	0.8711	0.9526	0.9826

Table 1. Meaning of the characteristic lengths of the same order of magnitude

Table 1 provides the meaning of the length $2l^*$, $3l^*$ and $5l^*$, characteristic lengths of the same order of magnitude as l^*.

The influence of Bi_t in the case of adiabatic end can be appreciated in fig. 5, which represents the temperature profile at r=0 and r=R of two relatively long fins of Bi_t = 2 and 0.5, defined by the values:

Pin fin 3:

$k_3 = 50$, $h_3 = 2000$,
$L_3 = 0.1060 = 3\,l^*$, $R_3 = 0.1$,
$\theta_{b,3} = 20ºC$, $\theta_{ref,3} = 10ºC$,
$Bi_{t,3} = h_3R_3/2k_3 = 2$, $l^*_3 = (k_3R_3/2h_3)^{1/2} = 0.03533$,

Pin fin 4:

$k_4 = 50$, $h_4 = 500$,
$L_4 = 0.212$, $= 3\,l^*$, $R_4 = 0.1$,
$\theta_{b,4} = 20ºC$, $\theta_{ref,4} = 10ºC$,
$Bi_{t,4} = h_4R_4/2k_4 = 0.5$, $l^*_4 = (k_4R_4/2h_4)^{1/2} = 0.0707$,

The Bi_t influence in fins of convective end can be seen in Fig. 6, which represents the temperature profile at r=0 and r=R of three fins different L/l^* values defined as follow:

Pin fin 5:

$k_5 = 50$, $h_5 = 2000$,
$L_5 = 0.03535$, $= l^*$, $R_5 = 0.1$,
$\theta_{b,5} = 50ºC$, $\theta_{ref,5} = 20ºC$,
$Bi_{t,5} = h_5R_5/k_5 = 2$, $l^*_5 = (k_5R_5/2h_5)^{1/2} = 0.03535$,

Pin fin 6:

$k_6 = 50$, $h_6 = 2000$,
$L_6 = 0.0707 = 2l^*$, $R_6 = 0.1$,
$\theta_{b,6} = 50ºC$, $\theta_{ref,6} = 20ºC$,
$Bi_{t,6} = h_6R_6/2k_6 = 2$, $l^*_6 = (k_6R_6/2h_6)^{1/2} = 0.03535$,

Pin fin 7:

$k_7 = 50$, $h_7 = 500$,
$L_7 = 0.03535 = 0.5l^*$, $R_7 = 0.1$,
$\theta_{b,7} = 50ºC$, $\theta_{ref,7} = 20ºC$,
$Bi_{t,7} = h_7R_7/2k_7 = 0.5$, $l^*_7 = (k_7R_7/2h_7)^{1/2} = 0.0707$,

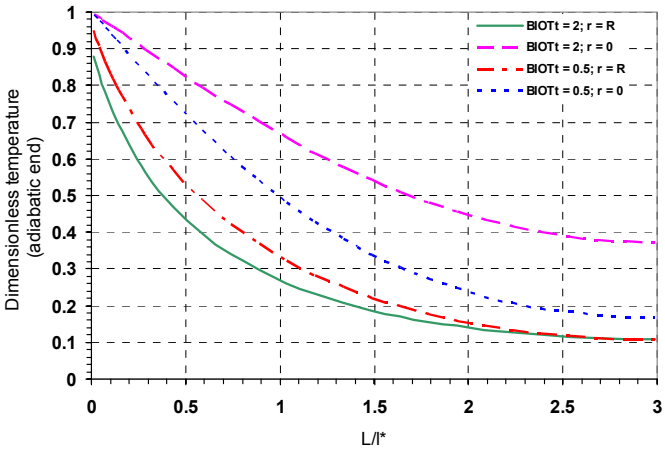

Fig. 5. Dimensionless temperature profiles for different Bi_t (adiabatic end)

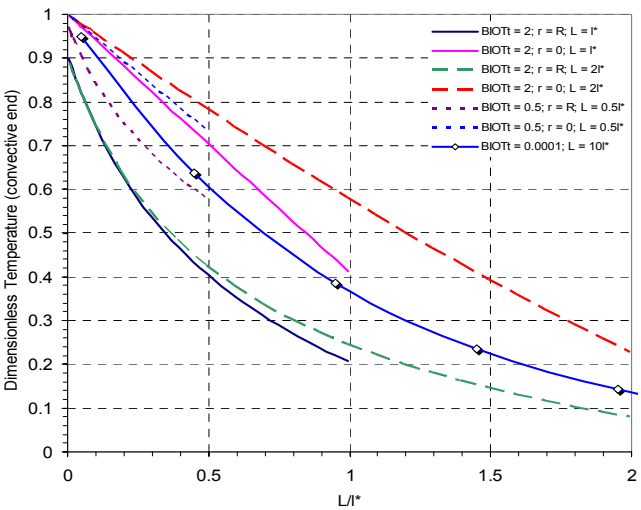

Fig. 6. Dimensionless temperature profiles for different Bi_t (convective end)

For comparison, Fig. 6 includes the typical profile of a long pin fin with a small Biot, Bi_t. = 0.0001. It is clear that the influence is higher as Bi_t increases.

5 The efficiency

The ratio of the actual heat loss to the maximum possible heat loss is termed the fin efficiency, η. We will use this extended and general purpose dimensionless coefficient to compare the performance of pin fins, with adiabatic and convective end, as a function of the ratio L/l^*.

Fig. 7 shows four efficiency curves. The (two) upper curves (very close to one another), are associated to adiabatic ends while the other two are associated to convective ends. These efficiencies have been derived by using a 2-D network model even if these same results can be obtained analytically from 1-D or 2-D assumptions due to the small value of Bi_t.

Fig. 7. Efficiency curves for small Bi_t

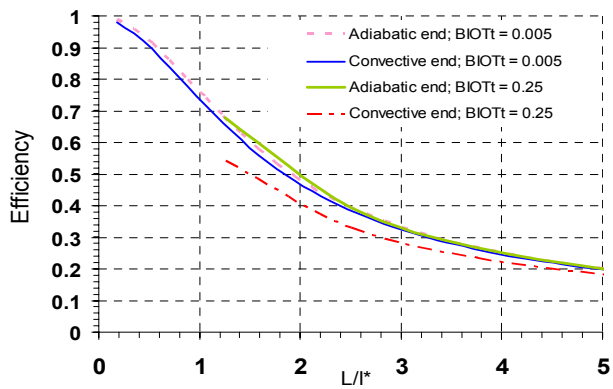

Fig. 8. Efficiency curves

The two upper curves, as expected, are universal because their Bi_t are very small (0.001 and 0.002). In contrast, the fins of convective end have an efficiency that depends on Bi_t according to the above conclusions 0.001 and 0.002, for the upper and lower curve, respectively.

Fig. 8 shows the efficiency of four fin pins (two of adiabatic and two of convective end) of Bi_t = 0.005 and 0.25. As expected, the only pin that presents different values of efficiency is that of convective end and Bi_t = 0.25. Fig. 9 shows the error of using the universal efficiency curve (associated to very small Bi_t or to pin fin with adiabatic end) for pin fins of convective end. $Biot_t$ numbers of these curves are very small, so that 1D approximation has been used.

As can be seen, the higher the Bi_t, the larger the error, which, in turn, varies with the ratio L/l^*. Bi_t = 0.02174 represents a limit value for which the efficiencies of pin fin of adiabatic and convective end may be approached with an error less than 5 %. It is interesting to note that this critical value has nothing to do with the typical Bi = 0.108 common in the lump model 1-D conduction.

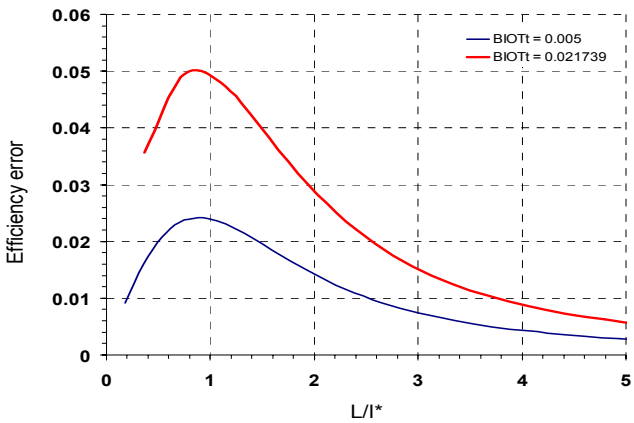

Fig. 9. Errors in the efficiency curves

Table 2 resumes the results of efficiency of 2-D pin fins for different Bi_t numbers obtained numerically by a network model. For comparison, the table includes the efficiency for the same fins, under the assumption of 1-D conduction, obtained both analytically and numerically (nearly closed one another). Errors obtained by using 1-D models for these fins are listed in the last line.

6 Conclusions

Discriminated dimensional analysis leads to and justifies the already known dimensionless numbers for cylindrical 1-D and 2-D pin fins with adiabatic or convective end and isothermal base as boundary conditions. More particularly, the existence of a characteristic length with a clear significance is proved whatever the mentioned boundary conditions. The dependence of efficiency (as the basis performance parameter for these fins) on these numbers is represented by graphs and a critical value for a transverse Biot number that approximates 1-D and 2-D fins with convective end is derived. This value is quite different from that used in 1-D lumped models due to the convection on the fin.

Bi_t	0.02174	0.04348	0.1087	0.2174	1.087	10.87
Analitical and numerical	0.745	0.7251	0.686	0.648	0.5185	0.2838
Numerical 2D	0.7407	0.717	0.671	0.6231	0.4747	0.2438
Error	$4.5\cdot10^{-3}$	$8.1\cdot10^{-3}$	$1.5\cdot10^{-3}$	$2.5\cdot10^{-3}$	$33.8\cdot10^{-3}$	$40.1\cdot10^{-3}$

Table 2. Efficiencies for 1-D and 2-D pin fins with convective end

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