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A Revision of the Classical Reynolds Numbers for Different Types of Flow Following the Theory of Discriminated Dimensional Analysis

MADRID GARCÍA C.N. and ALHAMA, F.
Department of Applied Physics
Polytechnic University of Cartagena
ETSII, Campus Muralla del Mar, Cartagena
SPAIN
paco.alhama@upct.es http://www.upct.es

Abstract: - The dimensionless character of classical Reynolds numbers currently used in the fluid flow boundary layer literature for plates and internal or external flows in circular pipes are here revised from the perspective of discriminate dimensional analysis. On the one hand, this perspective leads to an unambiguous form for the group of quantities that take part in what we will call the "discriminated Reynolds number", a new number that reduces to the classical Reynolds number in some geometries. On the other hand it is demonstrated that the application of discriminated dimensional analysis to boundary layer problems leads to monomials with a clear physical significance, i.e. the ratio "inertia forces/viscous forces", an interpretation that is not always correct for the classical Reynolds number.

Key-Words: - Classical Reynolds numbers, discriminated Reynolds number, discriminated dimensional analysis, inertia/viscous forces.

1 Introduction

In boundary layer fluid flow processes, the Reynolds number (Re hereinafter) is referred to in research literature and in most of text books as a typical dimensionless number that (approximately) characterizes the flow.

At low Re the flow is laminar, while at high Re the flow becomes turbulent [1-3]. The general expression for Re is

$$\mathrm{Re} = v^{*} l^{*}/\nu \qquad (1)$$

in which v^* (ms^{-1}) and l^* (m) denote characteristic velocity and length of the problem and ν (m^2s^{-1}) is the kinematics viscosity (or momentum diffusivity) of the fluid. Also, many text-books give Re the significance

Re ≡ ratio of inertial to viscous forces.

since the order of magnitude of these forces per unity of mass of fluid is v^2/l and $\nu v/l^2$, respectively.

The combination of quantities v^*, l^* and ν in Re arises from the immediate application of the dimensional analysis theory (or Buckingham pi theorem [4]) which consists of deducing the dimensionless independent groups of variables that can be formed from the relevant list of variables of the problem.

In Mechanics, the dimensional equation of these groups, named pi monomials, is $L^0\ M^0\ T^0$, where $\{L,M,T\}$ (length, mass and time), are the dimensions of the fundamental quantities that form part of the (dimensional) basis.

The most serious limitation for using dimensional analysis to provide some fundamental a priori information of the problem, is that it requires to know beforehand the exact variables that influence the problem, that is, it requires a thorough physical understanding of the phenomenon under study.

If the mathematical model of the problem is known, dimensionless groups can also be derived by manipulation of the differential equation. To this end, each dependent or independent variable of the equation is referred to other quantity of the problem of the same nature and finite value. The coefficients of the resulting differential equation would be the dimensionless groups that play a real role in the solution.

The aim of this paper is to demonstrate that classical dimensional analysis [5,6] leads, in general, to erroneous Re numbers that neither have physical significance (ratios of forces) nor play an independent role in the solution. In contrast, and using the same routine procedure as classical dimensional analysis, discriminate dimensional analysis leads to true Re that are really dimensionless and play a definitive role in the solution.

2 Classical dimensional analysis versus discriminated dimensional analysis

While classical dimensional analysis (CDA) is a scalar theory, i.e., it does not distinguish the vectorial character of most of the quantities and physical properties of materials, discriminated dimensional analysis (DDA) assumes this fundamental character. For CDA, lengths, velocities, forces, viscosity and other physical quantities are dimensionally independent (between them) if they are connected to different spatial directions.

In Mechanics, for example, the immediate consequence of this assumption is that the classical dimensional basis of the theory, $\{L,M,T\}$, becomes $\{L_x,L_y,L_z,M,T\}$ in the rectangular geometry. The increasing of the number of quantities in the basis (this number is called the multiplicity of the basis [5]) deals, in general, to a less number of dimensionless groups if the problem has the same number of variables.

Let us study the flow over a flat plate, figure 1. On the above basis, we take the length l_o in the direction of L_x, the velocity v_o in the direction of L_x, and the kinematics viscosity that recognizes the momentum diffusivity in the direction of L_z.

The discriminate dimensional equations of these quantities are:

$$[l_o] = L_x$$

$$[v_o] = L_x T^{-1}$$

$$[\nu] = L_z^2 T^{-1}$$

The grouping of these variables in the form of Reynolds number, $l_o v_o/\nu$, is not a dimensionless number since

$$[l_o v_o/\nu] = L_x^2/L_z^2$$

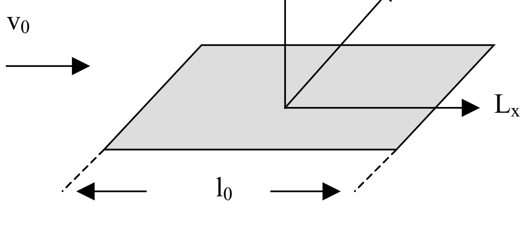

Fig. 1. Laminar flow over a flat plate

For cylindrical geometry, which is commonly used for internal and external (cross) flows, different discriminate basis would be chosen. Among others:

$\{L_r,L_s,L_z,M,T\}$

$\{L_r,\phi,L_z,M,T\}$

$\{S_z,L_s,L_z,M,T\}$

$\{L_r,S_r,L_z,M,T\}$ ...

where ϕ denotes the angle (rad), S_z denotes the surface normal to axial direction, and S_r the lateral surface of the cylinder.

The concept of discrimination is not only limited to spatial directions. Other physical considerations could be included within this concept. In the broad sense other types of discrimination that would increase the number of dimensions of the basis can be adopted. For example, mass may be used to account for inertia effects or to account for the amount of matter, which implies that two mass quantities must be recovered in the basis. Heat conduction is a kind of energy transport that has nothing to do with mass, so that, if simultaneous transport of heat and mass occurs and there is no conversion between mechanic and thermal energies (dissipations effects neglected), both Q and M will be part of the dimensional basis [6]. In addition, other quantities, such as surfaces and angles, could be included in the basis if it is convenient for the particular problem.

The only requirement, common to CDA and DDA, as regards the dimensional basis, is that the adopted basis has to be complete and the dimensions that contains independent [5].

3 The classical Reynolds number and its determination

It was Osborn Reynolds himself [8] who in the 19th-century established a dimensionless parameter, now called Reynolds, to distinguish the type of flow, laminar or turbulent, in a closed conduit. Later, this pure number was subsequently applied to other types of flow that are completely enclose or that involved a moving object completely immersed in a fluid.

The reasoning of Reynolds around his parameter were not made using the dimensional analysis theory but they came from the balance of the existing forces (inertial and viscous). This is a correct reasoning as long as the lengths for which the forces are evaluated were also correct. As we shall see later, these lengths can or cannot be of the same order of magnitude according to the geometry of the problem. The fact that Re was dimensionless (for the CDA theory), as is be expected, perhaps validated it in the eyes of the scientific community. Schlichting [9], for example, deduces Re by a pure (classical) dimensional reasoning.

According to the type of flow, figure 2, Re numbers ($v^* l^*/\nu$) are currently defined as:

Flat plate: $Re_L = v_\infty L/\nu$

External flow in pipes: $Re_D = v_\infty D/\nu$

Internal duct flow: $Re_D = v_\infty D/\nu$

The subscript L or D denotes the characteristic length (l^*) chosen in the definition of Re, and v_∞ is the non-disturbed fluid velocity.

Since there only exists a finite length for each one of the flows, pure (classical) dimensional reasoning (as that of Schlichting for flat plates) led to the above Re for external flow in pipes and internal duct flow.

4 The discriminate Re numbers

4.1 The dimensional basis

As mentioned above, discriminated dimensional analysis distinguishes as dimensionally independent the different spatial coordinates for each geometry.

Regardless of the type of flow, we will adopt a spatial coordinate connected to the direction of the fluid velocity (L_s) and other spatial coordinate connected to the normal to the sliding (viscous) surfaces (L_r). The third coordinate (L_z) which is necessary to express the dimensional equations of the variables of the problem, is normal to the two already defined, figure 2.

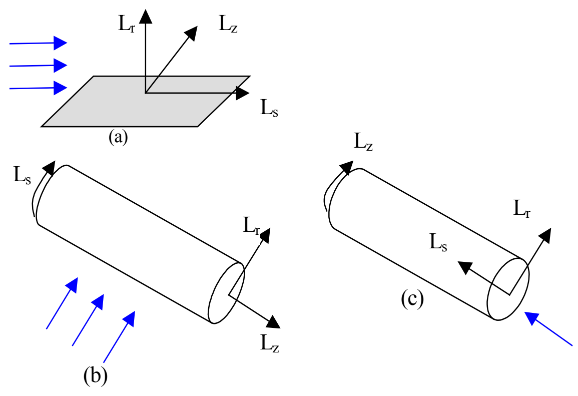

Fig. 2. Types of flow: a) flat plates, b)external flows in pipes, c) internal flow duct

Taking these spatial directions, the dimensional basis for the three problems will be

$\{L_r, L_s, L_z, M, T\}$

4.2 The relevant list of variables

The characteristic quantities that make up the relevant list of variables are:

(i) Flat Plate:

L, the length of the plate (or the length in the velocity direction where the force balance is established; the length normal to L that defines this region is unknown),

v_∞, the velocity of the fluid far from the boundary layer, and

ν, (or μ and ρ), the kinematics viscosity of the fluid (or the dynamic viscosity and the density of the fluid, separately)

(ii) External cross flow in (circular) pipes:

s, the length of arc that that defines the wet perimeter (s is one of the lengths that defines the region where the force balance is established, the other length that defines this region is unknown)

v_∞, as in flat plate,

ν, (or μ and ρ), as in flat plate,

(iii) Internal flow duct (large ducts):

D, the diameter of the pipe (also the transversal region where the force balance is made; the axial length of this region is unknown),

v_∞, as in flat plate,

ν, (or μ and ρ), as in flat plate,
(the axial length of duct is irrelevant in large ducts)

4.3 Dimensional equations of the variables

The dimensional equations of these quantities (note that the dimensional equation of μ come from its definition through the Newton law of viscosity, $F = \mu S (\partial v/\partial n)$ are:

(i) Flat Plate:

$[L] = L_s$

$[v_\infty] = L_s T^{-1}$

$[\rho] = M L_s^{-1} L_r^{-1} L_z^{-1}$

$[\mu] = M L_r L_s^{-1} L_z^{-1} T^{-1}$

$[\nu] = L_r^2 T^{-1}$

(ii) External flow in (circular) pipes:

$[s] = L_s$

$[v_\infty] = L_s\, T^{-1}$

$[\rho] = M\, L_r^{-1} L_s^{-1} L_z^{-1}$

$[\mu] = M L_r\, L_s^{-1} L_z^{-1} T^{-1}$

$[\nu] = L_r^{2} T^{-1}$

(iii) Internal flow duct (large ducts):

$[D] = L_r$

$[v_\infty] = L_s\, T^{-1}$

$[\rho] = M\, L_r^{-1} L_s^{-1} L_z^{-1}$

$[\mu] = M L_r\, L_s^{-1} L_z^{-1} T^{-1}$

$[\nu] = L_r^{2} T^{-1}$

Table 1 resumes the dimensional exponents of these variables according to in their respective basis.

4.4 The discriminate dimensionless groups

(i) Flat plate. From the variables L, v_∞ and ν (which substitute the pair μ and ρ) and their dimensional exponents of Table 1 a), the application of Buckingham pi theorem does not provide any dimensionless monomial. In consequence, no type of Re number appears in the solution.

Everything dimensional analysis contributes with this Table is the order of magnitude of δ, a "hidden quantity", in the radial direction, of dimension $[\delta] = L_r$, which undoubtedly is the boundary layer thickness that limits the region where the force balance applies. It is straightforward to obtain the order of magnitude of this quantity

$$\delta \sim (\nu L/v_\infty)^{1/2}$$

or, in terms of Re_L

$$\delta \sim L\, (Re_L)^{-1/2}$$

In this way, DDA does not lead to the classical Re but, instead, to a new one which we could define, for example, as

$$Re^d = (v_\infty L \nu /)(\delta/L)^2 = Re_L(\delta/L)^2$$

Due to the real dimensionless character of Re^d, the classical Re_L is related to the following meaning or order of magnitude

$$Re_L \sim (L/\delta)^2 >>1$$

the slenderness of the boundary layer region [10].

	L	v_∞	ρ	μ	ν
L_r			-1	1	2
L_s	1	1	-1	-1	
L_z			-1	-1	
M			1	1	
T		-1		-1	-1

a) Flat plate

	s	v_∞	ρ	μ	ν
L_r			-1	1	2
L_s	1	1	-1	-1	
L_z			-1	-1	
M			1	1	
T		-1		-1	-1

b) External flow in circular pipe

	D	v_∞	ρ	μ	ν
L_r	1		-1	-1	2
L_s		1	-1	-1	
L_z			-1	1	
M			1	1	
T		-1		-1	-1

c) Internal duct flow

Table 1. Dimensional exponents of the variables

(ii) External flows in circular pipes. From the variables s, v_∞ and ν and their exponents of Table 1 b), the application of Buckingham pi theorem does not provide dimensionless monomials but, again, a hidden length appears in the radial direction, $[\delta] = L_r$, with the same meaning as the former case, the boundary layer thickness.

$$\delta \sim (\nu s/v_\infty)^{1/2}$$

or, in terms of Re_s

$$\delta \sim s\, (Re_s)^{-1/2}$$

The Re^d provided by DDA is

$$Re^d = (v_\infty s/\nu)(\delta/s)^2 = Re_s\,(\delta/s)^2$$

and the order of magnitude of Re_s is

$$Re \sim (s/\delta)^2 >>1$$

again, the slenderness of the boundary layer region.

(iii) Internal duct flows. Under the hypothesis of large ducts the variable L (real length of the duct) is not relevant. The list of variables of the problem is D, v_∞ and ν and Table 1 c) of the dimensional exponents does not provide any dimensionless monomial.

The hidden length of this problem has the axial direction, $[l^*] = L_s$. The order of magnitude of l^* is

$$l^* = (v_\infty D^2/\nu)$$

l^* is the stretch of duct where the boundary layer is developed ($l^*<<L$). Obviously, the boundary layer thickness connected to l^* is R. The new discriminated dimensionless Re^d and its connection to the classical, Re_D, is:

$$Re^d = v_\infty D^2/(\nu l^*) = Re_D(D/l^*)$$

the meaning of Re_D also being the slenderness of boundary layer,

$$Re \sim l^*/D.$$

For the internal ducts flows with $L<l^*$, it is the length L which determines the thickness of the boundary layer, δ^*, and not D ($\delta^*<D$).

The list of variables is now L, v_∞ and ν, and the hidden quantity has the radial direction. From Table 2,

$$\delta^* = (L\nu/v_\infty)^{1/2}$$

The discriminated dimensionless, Re^d, and its connection to the classical, Re_D, is

$$Re^d = v_\infty\delta^2/(\nu L) = Re_D(\delta/L)^2$$

Again, the meaning of Re_D is the slenderness of boundary layer

$$Re_L \sim (L/\delta)^2.$$

a region of thickness in radial direction is less than the Radii.

	L	v_∞	ρ	μ	ν
L_r			-1	-1	2
L_s	1	1	-1	-1	
L_z			-1	1	
M			1	1	
T		-1		-1	-1

Table 2. Dimensional exponents of the variables for internal duct flow in short ducts

5 The meaning of dimensionless discriminated Reynolds numbers

The ratio of inertia forces to viscous forces per unity of mass for each one of the three flow configurations, within the boundary layer region confined by the two above mentioned lengths, yields the following expressions.

Flat plate:

f_i (inertia forces) = $\rho\, v_\infty^2/L$

f_v (viscous forces) = $\mu\, v_\infty/\delta^2$

$$f_i/f_v = (v_\infty\delta^2)/(\nu L) = Re_L(\delta/L)^2 = Re^d$$

Transversal flow in pipes:

f_i (inertia forces) = $\rho\, v_\infty^2/s$

f_v (viscous forces) = $\mu\, v_\infty/\delta^2$

$$f_i/f_v = (v_\infty\delta^2)/(\nu s) = Re_s\,(\delta/s)^2 = Re^d$$

Internal duct flow, $L>l^*$:

f_i (inertia forces) = $\rho\, v_\infty^2/ l^*$

f_v (viscous forces) = $\mu\, v_\infty/D^2$

$$f_i/f_v = (v_\infty D^2)/(\nu l^*) = Re_D(D/l^*) = Re^d$$

Internal duct flow, $L<l^*$:

f_i (inertia forces) = $\rho\, v_\infty^2/ L$

f_v (viscous forces) = $\mu\, v_\infty/\delta^{*2}$

$$f_i/f_v = (v_\infty\delta^{*2})/(\nu L) = Re_D(\delta^*/L)^2 = Re^d$$

When the $L \approx l^*$, the region of the boundary layer is confined by R and L (or δ and l^*), since $R \approx \delta$. Using these approximations the ratio of inertia forces to viscous forces provides the same results at the last two cases.

As is shown, the meaning of the discriminated dimensional Reynolds number is clear and does not depend on the flow configuration, an important conclusion that confirms that the good use of discriminated dimensional analysis, in contrast to classical dimensional analysis, leads to definitive monomials that play an unquestionable role in the solution.

6 Conclusions

Discriminated dimensional analysis is demonstrated to be a fundamental and fast technique which is capable of deriving fundamental information for the problem under study without the need to develop bothersome mathematical calculus.

For the flow regimes studied, in contrast to classical dimensional analysis, discriminated dimensional analysis leads to (discriminated) Reynolds numbers with a clear meaning common to the three types of flow: flow along flat plates, external flow in pipes and internal flow in ducts. This meaning is the ratio of inertia to viscous forces within the region where the boundary layer develops.

The connection between classical and discriminated Reynolds is immediately shown and, from this connection, a geometrical meaning can be attributed to the classical dimensionless numbers.

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