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Analytical solution of one-dimensional transient heat conduction problem in thick metal slab

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ABSTRACT:

The present work represents the analytical solution of one-dimensional transient heat conduction problem in thick metal slab. This work deals with a large slab of 0.3 m thick steel armour plate initially at a uniform temperature of 710 ^{o}C. One surface is maintained at 710 ^{o}C while air is blown over the other surface which gives rise to an average heat-transfer coefficient of 113 $\frac{w}{m^2C}$. The temperature of air is T_∞ = 318 ^{o}C. The surface temperature and the distribution after 0.5 , 0.9 and 1.0 hour had elapsed were measured.

In this case, the analytical solution is possible, in principle, and the mechanics of obtaining the solution is presented herein.

Key words: Heat transfer, large slab, analytical solution.

1. INTRODUCTION

Heat transfer and temperature distribution in the slabs made of steel or other material play an important role in thermal applications. Slabs in casting mould, or slabs in buildings structures or slabs in engines (Internal Combustion) have sometimes complex geometry; which make the use of numerical method easier for solving heat transfer problems rather than the complexity of the analytical solution associated with there practical engineering applications also non uniform boundary conditions, time dependent boundary conditions and temperature dependent properties.

In some cases, analytical solutions are possible, in principle, but the mechanics of obtaining the solution may be much more difficult than the task of solving the problem numerically as present here.

Heat transfer in slabs has been the subject of investigations for many researchers [1-6]. In these studies i.e A.Paul et al [1] investigated the local heat transfer (Flux) in a slab caster, to determine the heat flux and temperature distribution through the mould wall. By experimental rig using thermo couples then they developed a mathematical model for mould.

Brian et al [2] made a model of heat transfer between slabs-on-grade and the ground to evaluate the dynamic behavior to evaluate the coefficient of the equations which governed these models to improve buildings heating and cooling loads to optimize energy use.

Meng et al [3] adopted a model of one dimensional heat transfer and solidification of the continuous casting of steel slabs. This model besides it is one dimensional it is also transient and therefore they use the finite difference calculations of heat conduction within the solidifying steel shell coupled with two dimensional steady state with the mould wall slabs.

Belet et al [4] used the finite element and boundary element techniques to describe the temperature distribution and heat flux through slabs of continuous casting process.

Richard et al [5] developed and verified a fundamentally based model for low temperature radiant system which can be used within energy analysis programs to evaluate the temperature distribution and heat flux variation through one dimensional transient condition heat problems. They presented their work using Laplace transform techniques and also state space method.

In this work the one dimensional heat problem which is time dependent is solved analytically, and the last formula obtained is a new formula.

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2. ANALYTICAL SOLUTION

Applying the transient one-dimensional heat conduction Equ. and using the steel slab stated as shown:

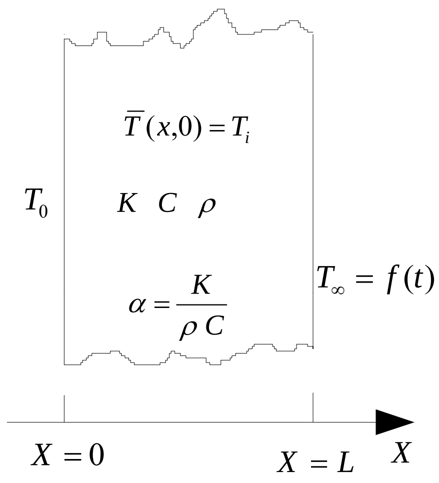

Figure 1: schematic representation of steel slab

$$\frac{\partial T(x,t)}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{1}$$

Where $\alpha = \dfrac{K}{\rho.C}$ = thermal diffusivity.

Subject to the boundary conditions:

B.C.(1) $T(0,t) = T_0$ …(2a)

B.C.(2) $-K\dfrac{\partial T(L,t)}{\partial x} = h[T(L,t) - f(t)]$ (2b)

Initial conduction $T(x,0) = T_i$ …(2c)

Let $\overline{T}(x,t) = T(x,t) - T_0$ (3)

Then Equ.(1) and its B.Cs. become:

$$\frac{\partial \overline{T}}{\partial t} = \alpha \frac{\partial^2 \overline{T}}{\partial x^2} \tag{4}$$

$$\overline{T}(0,t) = 0 \tag{5}$$

$$-K\frac{\partial \overline{T}(L,t)}{\partial x} = h[\overline{T}(L,t) + T_0 - f(t)] \tag{6}$$

$$\overline{T}(x,0) = T_i - T_o \tag{7}$$

$$\overline{T}(0,t) = 0 \tag{8}$$

$$-\frac{\partial \overline{T}(L,t)}{\partial t} = \frac{h}{K}\overline{T}(L,t) + \frac{h}{K}[T_0 - f(t)] \tag{9}$$

Let $\overline{T}(x,t) = \xi(x,t) + w(x,t)$ (10)

Then Equ. (4) reduces to:

$$\frac{\partial \xi}{\partial t} - \alpha\frac{\partial^2 \xi}{\partial x^2} = -\frac{\partial w}{\partial t} + \alpha\frac{\partial^2 w}{\partial x^2} \tag{11}$$

The B.C. at x=0 becomes

$$\xi(0,t) = -w(0,t) \tag{12}$$

Also the B.C. at x=L reduces to

$$-\frac{\partial \xi(L,t)}{\partial x} - \frac{h}{k}\xi(L,t) = \frac{\partial w(L,t)}{\partial x} + \frac{h}{k}w(L,t) + \frac{h}{k}[T_0 - f(t)] \tag{13}$$

now assume that:

$$w(0,t) = 0 \tag{14}$$

then $\xi(0,t) = 0$ (15)

Now separate w,ξ in Equ. (13) such that:

$$\frac{\partial w(L,t)}{\partial x} + \frac{h}{k}w(L,t) = -\frac{h}{k}[T_0 - f(t)] \tag{16}$$

and

$$\frac{\partial \xi(L,t)}{dx} = -\frac{h}{k}\xi(L,t) \tag{17}$$

The solution of Equ. (16) is:

$w(x,t) = [T_0 - f(t)]\,(A + Bx)$ at x=L (18)

Where A and B are constants

Since $w(0,t) = 0$, then A=0 since $T_0 - f(t) \neq 0$

Then Equ. (18) becomes:

$w(x,t) = [T_0 - f(t)]\,Bx$ at x = L (19)

Therefore;

$$\frac{\partial w(x,t)}{\partial x} = [T_0 - f(t)]\,B \tag{20}$$

Substitute Equ.s. (19) and (20) into Equ. (16) at x=L to obtain:

$$[T_0 - f(t)]B + \frac{h}{k}[T_0 - f(t)]BL = -\frac{h}{k}[T_0 - f(t)] \tag{21}$$

Solving for the constant B then:

$$B = -\frac{\frac{h}{k}}{1 + \frac{hL}{k}} \tag{22}$$

Substitute Equ. (22) into Equ. (19):

$$w(x,t) = [T_0 - f(t)]\left(\frac{-\frac{hL}{k}}{1+\frac{hL}{k}}\right)\frac{x}{L} \tag{23}$$

Differentiate Equ. (23) with respect to t and x then

$$\frac{\partial w}{\partial t} = f'(t)\frac{\frac{hL}{K}}{1+\frac{hL}{K}}\cdot\frac{x}{L} \tag{24}$$

and:

$$\frac{\partial^2 w(x,t)}{\partial x^2} = 0 \tag{25}$$

Substitute Equ. (24) and (25) into Equ. (11):

$$\frac{\partial \xi}{\partial t} - \alpha\frac{\partial^2 \xi}{\partial x^2} = -\frac{\frac{hL}{K}}{1+\frac{hL}{K}} f'(t)\frac{x}{L} \tag{26}$$

Equ. 26 can be solved by the method of separation of variable as follows:

$$\frac{\partial \xi_c}{\partial t} - \alpha\frac{\partial^2 \xi_c}{\partial x^2} = 0 \tag{27}$$

Where ξ_c is the complemntary function of variable ξ_c
To solve Equ. (27) by separation of variable method, let:

$$\xi_c = g(t)\,.\,Y(x) \tag{28}$$

Substitute Equ. (28) into Equ. (27) to get:

$$\frac{g'(t)}{\alpha\, g(t)} = \frac{y''(x)}{y(x)} = -\lambda^2 \tag{29}$$

The solution of Equ. (29) is expressed as:

$$\xi_c = e^{-\lambda^2\alpha t}[c_1\cos(\lambda x) + c_2\sin(\lambda x)] \tag{30}$$

Substitute Equ. (15) into Equ. (30) then:

$$-\lambda x e^{-\lambda^2\alpha t}\;[c_1.1 - c_2.0] = 0$$

therefore, $c_1 = 0$
Differentiate Equ. (30) with respect to x:

$$\frac{\partial \xi_c}{\partial x} = -\lambda e^{-\lambda^2\alpha t}[c_1\sin(\lambda x) - c_2\cos(\lambda x)] \tag{31}$$

Therefore:

$$\xi_c = e^{-\lambda^2\alpha t}c_2\sin(\lambda x) \tag{32a}$$

and,

$$\frac{\partial \xi_c}{\partial x} = \lambda e^{-\lambda^2\alpha t}[c_2\cos(\lambda x)] \tag{32b}$$

Substitute Equ. (32a) and (32b) into Equ. (17)

$$-\frac{h}{k}e^{-\lambda^2\alpha t}c_2\sin(\lambda L) = \lambda e^{-\lambda^2\alpha t}c_2.\cos(\lambda L) \tag{33}$$

Where: $\lambda L = \lambda_1.L, \lambda_2.L, ..., \lambda_n.L$

Therefore:

$$\tan(\lambda_n L) = \frac{-K\lambda_n L}{hL} \tag{34}$$

or

$$\cot(\lambda_n L) = -\frac{hL}{K\lambda_n L} \tag{35}$$

To find the values of λ_n the graphs of $\cot(\lambda_n)$ and $\frac{-h}{K\lambda L}$ verses λ_n are drawn as shown in Fig.2

Let $$\xi(x,y) = \sum_{n=1}^{\infty} Q_n(t)\sin(\lambda_n L\frac{x}{L}) \tag{36}$$

Expand $$\frac{x}{L} = \sum_{n=1}^{\infty} P_n\sin(\lambda_n x) \tag{37}$$

$$\int_0^L \frac{x}{L}\sin(\lambda_n x)dx = P_n\int_0^L \sin^2(\lambda_n x)dx \tag{38}$$

Then let

$$\eta = \lambda_n x \qquad\qquad dx = \frac{d\eta}{\lambda_n}$$

Therefore, Equ. (38) becomes:

$$\frac{1}{\lambda_n L}\int_0^{\lambda_n L}\eta\sin(\eta)d\eta = P_n\int_0^{\lambda_n L}\eta\sin^2(\eta)d\eta \tag{39}$$

$$\frac{1}{\lambda_n L}[\sin(\eta) - \cos(\eta)]_0^{\lambda_n L} = P_n[\frac{\eta}{2} - \frac{1}{4}\sin 2\eta]_0^{\lambda_n L} \tag{40}$$

solve for P_n then :

$$P_n = \frac{\frac{1}{\lambda_n L}[\sin(\lambda_n L) - \lambda_n L\cos(\lambda_n L)]}{\frac{\lambda_n L}{2} - \frac{1}{2}\sin(\lambda_n L).\cos(\lambda_n L)} \tag{41}$$

But

$$\cos(\lambda_n L) = -\frac{h}{\lambda K}\sin(\lambda_n L)$$

Therefore,

$$P_n = \frac{\frac{1}{\lambda_n L}\sin(\lambda_n L)[1+\frac{Lh}{K}]}{\frac{\lambda_n L}{2}+\frac{h}{2\lambda_n K}\sin^2(\lambda_n L)} \quad (42)$$

multiply Equ. (42) by $\frac{2\lambda_n L}{2\lambda_n L}$ then:

$$P_n = \frac{2\sin(\lambda_n L)[1+\frac{hL}{K}]}{(\lambda_n L)^2+\frac{hL}{K}\sin^2\lambda_n L} \quad (43)$$

Refer to Equ.(26),(37) and (40) then the following relation is obtained: Substituting with the complete Equ. of $\xi(x,t)$:

$$\sum_{n=1}^{\infty}\dot{Q}_n(t)\sin(\lambda_n L\frac{x}{L})-\alpha\sum_{n=1}^{\infty}Q_n(t)(-\lambda_n^2)\sin(\lambda_n L\frac{x}{L}) = -\sum_{n=1}^{\infty}\frac{\frac{hL}{K}f'(t)}{1+\frac{hL}{K}}\frac{2\sin(\lambda_n L)[1+\frac{hL}{K}]}{(\lambda_n L)^2+\frac{hL}{K}\sin^2\lambda_n L}\sin(\lambda_n L\frac{x}{L}) \quad (44)$$

$$\dot{Q}_n(t)+\lambda_n^2\alpha Q_n(t) = -\frac{2\frac{hL}{K}f'(t)\sin(\lambda_n L)}{(\lambda_n L)^2+\frac{hL}{K}\sin^2\lambda_n L} \quad (45)$$

$$e^{\lambda_n^2\alpha t}(\dot{Q}_n(t)+\lambda_n^2\alpha Q_n(t)) = -\frac{2\frac{hL}{K}\sin(\lambda_n L)}{(\lambda_n L)^2+\frac{hL}{K}\sin^2\lambda_n L}e^{\lambda_n^2\alpha t}f'(t) \quad (46)$$

$$Q_n(t)e^{\lambda_n^2\alpha t} = c_1-\frac{2\frac{hL}{K}\sin(\lambda_n L)}{(\lambda_n L)^2+\frac{hL}{K}\sin^2\lambda_n L}\int e^{\lambda_n^2\alpha t}f'(t)dt \quad (47)$$

$$Q_n = c_1e^{-\lambda_n^2\alpha t}-\frac{2\frac{hL}{K}e^{-\lambda_n^2\alpha t}\sin(\lambda_n L)}{(\lambda_n L)^2+\frac{hL}{K}\sin^2\lambda_n L}\int e^{\lambda_n^2\alpha t}f'(t)dt \quad (48)$$

$$\xi(x,t) = \sum_{n=1}^{\infty}[c_{1n}e^{-\lambda_n^2\alpha t}-\frac{2\frac{hL}{K}e^{-\lambda_n^2\alpha t}\sin(\lambda_n L)}{(\lambda_n L)^2+\frac{hL}{K}\sin^2\lambda_n L}\int^{t}e^{\lambda_n^2\alpha t}f'(t)dt]\sin(\lambda_n x) \quad (49)$$

Noting that:

$$\bar{T} = \xi-\frac{[T_0-f(t)]\frac{hL}{K}}{1+\frac{hL}{K}}\frac{x}{L}$$

$$Then\ :(T_i-T_0)+[T_0-f(0)]\frac{\frac{hL}{K}}{1+\frac{hL}{K}}\frac{x}{L} = \sum_{n=1}^{\infty}[c_{1n}-\frac{2\frac{hL}{K}\sin(\lambda_n L)}{(\lambda_n L)^2+\frac{hL}{K}\sin^2\lambda_n L}\int^{0}e^{\lambda_n^2\alpha t}f'(t)dt]\sin(\lambda_n x) \quad (50)$$

$$(T_i-T_0)\int_0^L\sin(\lambda_n x)dx+[T_0-F(0)]\frac{\frac{hL}{K}}{1+\frac{hL}{K}}\int_0^L\frac{x}{L}\sin(\lambda_n x)dx = [c_{1n}-\frac{2\frac{hL}{K}\sin(\lambda_n L)}{(\lambda_n L)^2+\frac{hL}{K}\sin^2\lambda_n L}\int e^{\lambda_n^2\alpha t}f'(t)dt]\int_0^L\sin^2(\lambda_n x)dx \quad (51)$$

$$\frac{(T_i-T_0)}{\lambda_n L}\cos(\lambda_n x)]_L^0+[T_0-F(0)]\frac{\frac{hL}{K}}{1+\frac{hL}{K}}\frac{1}{\lambda_n^2L^2}(\sin(\lambda_n L)-\lambda_n L\cos(\lambda_n L)) = [c_{1n}-\frac{2\frac{hL}{K}\sin(\lambda_n L)}{(\lambda_n L)^2+\frac{hL}{K}\sin^2\lambda_n L}\int^{0}e^{\lambda_n^2\alpha t}f'(t)dt]\frac{\lambda_n L-\cos(\lambda_n L)\sin(\lambda_n L)}{2\lambda_n L} \quad (52)$$

$$\frac{(T_i-T_0)}{\lambda_n L}[1-\cos(\lambda_n x)]+[T_0-f(0)]\frac{\frac{hL}{K}}{\lambda_n^2L^2}\sin(\lambda_n L) = \frac{(\lambda_n L)^2+\sin^2(\lambda_n L)\frac{hL}{K}}{2\lambda_n L}[c_{1n}-\frac{2\frac{hL}{K}\sin(\lambda_n L)}{(\lambda_n L)^2+\frac{hL}{K}\sin^2\lambda_n L}\int e^{\lambda_n^2\alpha t}f'(t)dt] \quad (53)$$

$$(T_i-T_0)[1-\cos(\lambda_n x)]+[T_0-f(0)]\frac{hL}{K}\frac{\sin(\lambda_n L)}{\lambda_n L} = \frac{(\lambda_n L)^2+\sin^2(\lambda_n L)\frac{hL}{K}}{2\lambda_n L}[c_{1n}-\frac{2\frac{hL}{K}\sin(\lambda_n L)}{2\lambda_n L}\int^{0}e^{\lambda_n^2\alpha t}f'(t)dt] \quad (54)$$

$$c_{1n} = \frac{(T_i-T_0)[1-\cos(\lambda_n x)]\frac{\frac{hL}{K}}{\lambda_n^2L^2}\sin(\lambda_n L)+\frac{hL}{K}\frac{\sin(\lambda_n L)}{\lambda_n L}\int e^{\lambda_n^2\alpha t}f'(t)dt}{\frac{(\lambda_n L)^2+\frac{hL}{K}\sin^2\lambda_n L}{2\lambda_n L}} \quad (55)$$

$$c_{1n} = \frac{(T_i-T_0)[1-\cos(\lambda_n x)]+[T_0-f(0)]\frac{hL}{K}\frac{\sin(\lambda_n L)}{\lambda_n L}}{\frac{(\lambda_n L)^2+\frac{hL}{K}\sin^2\lambda_n L}{2\lambda_n L}}+\frac{2\frac{hL}{K}\sin(\lambda_n L)}{(\lambda_n L)^2+\frac{hL}{K}\sin^2\lambda_n L}\int e^{\lambda_n^2\alpha t}f'(t)dt \quad (56)$$

$$\bar{T}(x,t) = -[T_0-f(0)]\frac{\frac{hL}{K}}{1+\frac{hL}{K}}\frac{x}{L} + \left(\begin{array}{l}\sum_{n=1}^{\infty}\frac{[2\lambda_n L(T_i-T_0)[1-\cos(\lambda_n x)]+2[T_0-f(0)]\frac{hL}{K}\sin(\lambda_n L)e^{-\lambda_n^2\alpha t}}{(\lambda_n L)^2+\frac{hL}{K}\sin^2\lambda_n L} \\ +\frac{2\frac{hL}{K}\sin(\lambda_n L)e^{-\lambda_n^2\alpha t}}{(\lambda_n L)^2+\frac{hL}{K}\sin^2\lambda_n L}\int^{0}e^{\lambda_n^2\alpha t}f'(t)dt \\ -\frac{2\frac{hL}{K}\sin(\lambda_n L)e^{-\lambda_n^2\alpha t}}{(\lambda_n L)^2+\frac{hL}{K}\sin^2\lambda_n L}\int^{t}e^{\lambda_n^2\alpha t}f'(t)dt.\end{array}\right)\sin\lambda_n x \quad (57)$$

$$\bar{T}(x,t) = T(x,t)-T_0 = -[T_0-f(0)]\frac{\frac{hL}{K}}{1+\frac{hL}{K}}\frac{x}{L} +2\sum_{n=1}^{\infty}\frac{e^{-\lambda_n^2\alpha t}}{(\lambda_n L)^2+\frac{hL}{K}\sin^2\lambda_n L}[\lambda_n L(T_i-T_0)[1-\cos(\lambda_n x) +[T_0-f(0)]\frac{hL}{K}\sin(\lambda_n L)-\frac{hL}{K}\sin(\lambda_n L)\int_0^t e^{\lambda_n^2\alpha t}f'(\eta)d\eta]\sin(\lambda_n x) \quad (58)$$

Hence the temperature distribution of the present work is:

$$\bar{T}(x,t) = T(x,t) - T_0 = -[T_0 - f(0)]\frac{\frac{hL}{K}}{1+\frac{hL}{K}}\frac{x}{L} \tag{59}$$

$$+2\sum_{n=1}^{\infty}\frac{e^{-\lambda_n{}^2\alpha t}}{(\lambda_n L)^2 + \frac{hL}{K}\sin^2\lambda_n L}[\lambda_n L(T_i - T_0)[1-\cos(\lambda_n x)]$$

$$+[T_0 - f(0)]\frac{hL}{K}\sin(\lambda_n L) - \frac{hL}{K}\sin(\lambda_n L)\int_0^t e^{\lambda_n{}^2\alpha t} f'(\eta)d\eta]\sin(\lambda_n x)$$

λ_n can be determined from figure 2 or calculated from Equ. (34) and (35).

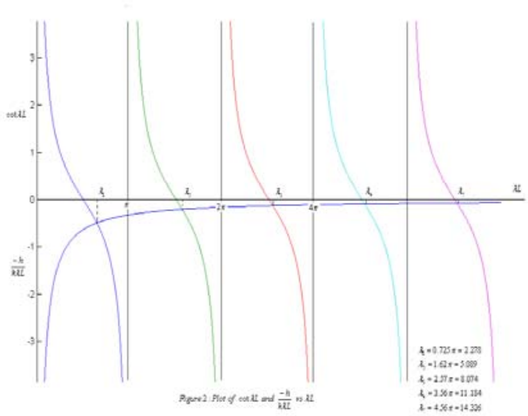

Figure 2: Plot of $\cot \lambda L$ and $\frac{-h}{k\lambda L}$ vs λL

3. Results and Discussion:

The derived temperature distribution relation as expressed in Equ. (59) was used to obtain the temperature distribution throughout the thick slab after 0.5, 0.9 and 1 hour.

Overall, the results of this work turned out as expected. It was confirmed that the analytical method employed could be used to give an accurate solution which satisfy the needs of the engineers and designers who always push for more efficient and economic design.

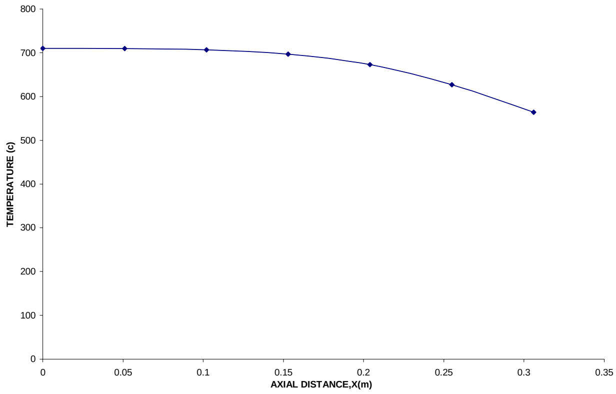

FIGURE -3- TEMPERATURE DISTRIBUTION AFTER 0.5 HOUR

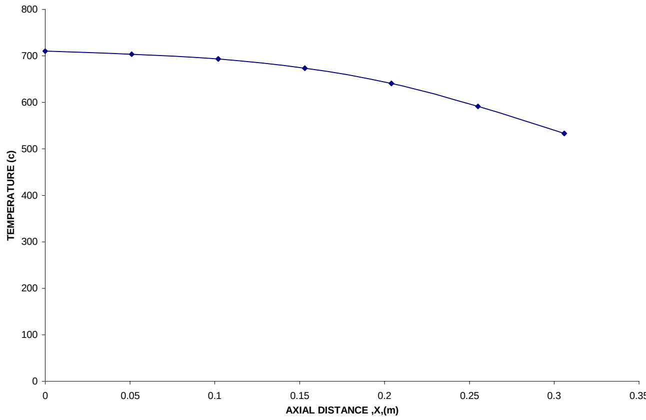

FIGURE-4- TEMPERATURE DISTRIBUTION IN THE SLAB AFTER 0.9 HOUR ELAPSED

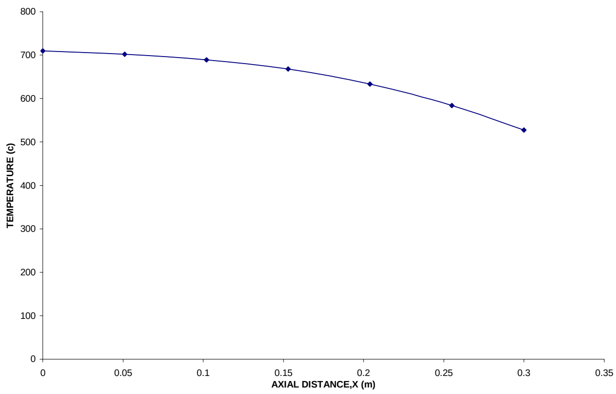

FIGURE-5- TEMPERATURE DISTRIBUTION IN THE SLAB AFTER ONE HOUR ELAPSED

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