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Exterior stationary Navier-Stokes flows in 3D with nonzero velocity at infinity: asymptotic behaviour of the velocity and its gradient.

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Abstract: We consider stationary incompressible Navier-Stokes flows in an exterior domain in $\mathbb{R}^3$. Under the assumption that the velocity at infinity is nonzero, we study how the velocity, its gradient and and the pressure behave far from the complement of the exterior domain.

Key Words: Stationary incompressible Navier-Stokes flows, exterior domains, asymptotic behaviour.

1 Introduction

When exterior flows with nonzero velocity at infinity are modeled by the stationary incompressible Navier-Stokes system, then usually the following boundary value problem arises:

$$-\Delta u + \tau \cdot D_1 u + \tau \cdot (u \cdot \nabla) u + \nabla \pi \;=\; f, \qquad (1)$$
$$\operatorname{div} u \;=\; 0 \quad \text{in } \mathbb{R}^3 \backslash \overline{\Omega},$$
$$u \,|\, \partial\Omega = b, \qquad |u(x)| \to 0 \quad \text{for } |x| \to \infty,$$

with $\tau \in (0,\infty)$ (Reynolds number), and $\Omega \subset \mathbb{R}^3$ a bounded open set. Note that the zero boundary conditions at infinity appear here because the original problem with nonzero conditions was transformed by a translation; see [1]. In fact, the transformed problem is easier to treat than the original one. We suppose that Ω has a connected complement and a compact Lipschitz boundary. Abbreviate $V^c := \mathbb{R}^3 \backslash V$ for $V \subset \mathbb{R}^3$, $B_r := \{y \in \mathbb{R}^3 \,:\, |y| < r\}$ and $\Omega_r := B_r \backslash \overline{\Omega}$ for $r \in (0,\infty)$. Then we assume that $f \in L^{6/5}(\overline{\Omega}^c)^3$, and that there are numbers γ, $S \in (0,\infty)$, $\sigma \in (4,\infty)$ such that $\overline{\Omega} \subset B_S$ and

$$|\, f(y)\,| \;\le\; \gamma \cdot |y|^{-\sigma} \quad \text{for } y \in B_S^c. \qquad (2)$$

In addition, we suppose that problem (1) admits a solution (u,π) with these properties:

$$u \in W^{2,\,6/5}_{loc}(\overline{\Omega}^c)^3, \;\; \pi \in W^{1,\,6/5}_{loc}(\overline{\Omega}^c), \qquad (3)$$
$$\nabla u \in L^2(\overline{\Omega}^c)^9,$$
$$R \cdot \int_{\partial B_1} |\, u(R\cdot x)\,|^2 \, do_x \;\to\; 0 \;\text{ for }\; R \to \infty,$$
$$u \,|\, \Omega_R \in L^2(\Omega_R)^3, \;\; \pi \,|\, \Omega_R \in L^2(\Omega_R),$$
$$\pi \,|\, B_R^c \in L^3(B_R^c) \quad \text{for } R \in (S,\infty).$$

Such a solution exists if the preceding assumptions on f and Ω are valid, and if $b \in H^{1/2}(\partial\Omega)^2$ and the flow of b through $\partial\Omega$ is small in the sense of [6, Theorem IX.4.1]; see [6, Theorem IX.4.1, IX.1.1, Lemma IX.1.1]. Note that the assumption $f \in L^{6/5}(\overline{\Omega}^c)^3$ implies that f belongs to the function space $\mathcal{D}_0^{-1,2}(\overline{\Omega}^c)$ from [6, Theorem IX.4.1]. Actually, according to the preceding references, this latter relation is the only condition which has to be imposed on f in order to obtain a solution of (1) with properties as stated in (3).

The domain Ω, the functions f, u and π, and the parameters τ, S and γ will be kept fixed throughout. We want to study the decay of u, ∇u

and π when $|x|$ tends to infinity. Before stating our main result in this respect, let us introduce some further notations. By the letter $\mathcal{C}$, we denote constants depending on Ω, S, τ, γ, σ and u. For $r > 0$, $x \in \mathbb{R}^3$, we write $B_r(x)$ for the open ball with radius r and center at x (hence $B_r = B_R$). Put for $x \in \mathbb{R}^3 \backslash \{0\}$,

$$\nu^{\alpha}_{\beta}(x,\tau) := |x|^{\alpha} \cdot (1 + \tau \cdot |x| - \tau \cdot x_1)^{\beta},$$
$$\nu^{\alpha}(x) := |x|^{\alpha} \quad \text{for } \alpha, \beta \in \mathbb{R}, \ \beta \neq 0;$$
$$\varrho(x,\tau) := |x|^{-2} + \tau^{1/2} \cdot \nu^{-3/2}_{-3/2}(x,\tau).$$

Let $V \subset \mathbb{R}^3$ be bounded and open, with $0 \in V$. For a function $v : V^c \mapsto \mathbb{R}$, and for $\alpha, \beta \in \mathbb{R}$ with $\beta \neq 0$, $\sigma \in \{ \nu^{\alpha}_{\beta}(\cdot ,\tau), \nu^{\alpha}, \varrho(\cdot ,\tau)^{-1} \}$, we put $\|v\|_{\infty,\sigma} := \sup\{ |v(x)| \cdot \sigma(x) : x \in V^c \}$.

It turned out that the norm $\| \ \|_{\infty, \nu^1_1(\cdot ,\tau)}$ is a good choice to measure the decay of u, and the norm $\| \ \|_{\infty, \varrho(\cdot ,\tau)^{-1}}$ is well adapted to characterize the asymptotic behaviour of ∇u. In fact, the ensuing theorem holds:

Theorem 1 *There is some $S_0 \in [S,\infty)$ with*

$$\| u \,|\, B^c_{S_0} \|_{\infty, \nu^1_1(\cdot ,\tau)} + \| \nabla u \,|\, B^c_{S_0} \|_{\infty, \varrho(\cdot ,\tau)^{-1}} + \| \pi \,|\, B^c_{S_0} \|_{\infty, \nu^2} < \infty.$$

Theorem 1 was shown in [3] for small values of τ; see [3, Theorem 1.2]. Here we do not impose a condition on τ. In this case, the relation $\| u \,|\, B^c_{S_0} \|_{\infty, \nu^1_1(\cdot ,\tau)} < \infty$ is new, whereas the results on ∇u and π may be read into [5] and [1]; also see [6, Section IX.8]. The aim of this article consists in establishing the preceding relation for u, and at the same time presenting a rather direct access to all the results of Theorem 1. It turned out that such an access was possible on the basis of [6, Theorem IX.7.1] (L^p-regularity of u near infinity) and some results from [7] (weighted pointwise estimates of convolutions of the Oseen fundamental solution). Theorem 1 plays an essential role in [2], so we think it is worthwhile to study its proof.

2 Proof of Theorem 1

In the following, we will use a result from [4], which states that for any $\beta \in (0,\infty)$, there is $C(\beta) > 0$ with

$$\int_{\partial B_r} (1 + |y| - y_1)^{-\beta} \, do_y \quad (4)$$
$$\leq C(\beta) \cdot r^{2-\min\{1,\beta\}} \cdot \sigma(r) \quad \text{for } r \in (0,\infty),$$

with $\sigma(r) := 1$ if $\beta \neq 1$, and $\sigma(r) := 1 + \ln(1+r)$ if $\beta = 1$. Moreover we will need [3, Lemma 4.8], which states that

$$|x|^{-1} \leq \mathcal{C} \cdot (1 + \tau \cdot |x| - \tau \cdot x_1)^{-1} \quad (5)$$

for $x \in B^c_S$. Assumption (2) implies in particular that $f \,|\, \overline{B_S}^c \in L^r(\overline{B_S}^c)^3$ for any $r \in [1,\infty)$. Therefore we know by [6, Theorem VIII.5.1] that $u \,|\, \overline{B_S}^c \in W^{2,r}_{loc}(\overline{B_S}^c)^3$, $\pi \,|\, \overline{B_S}^c \in W^{1,r}_{loc}(\overline{B_S}^c)$ for any $r \in [1,\infty)$.

Now fix some $T \in (S,\infty)$, for example, $T = 2 \cdot S$. It follows by Sobolev inequalities,

$$u \,|\, B^c_T \in C^1(B^c_T)^3, \quad \pi \,|\, B^c_T \in C^0(B^c_T). \quad (6)$$

Moreover, by [6, Theorem XI.7.1],

$$u \,|\, B^c_T \in L^r(B^c_T)^3 \quad \text{for } r \in (2,\infty], \quad (7)$$
$$\nabla u \,|\, B^c_T \in L^r(B^c_T)^9 \quad \text{for } r \in (4/3, \infty].$$

Define

$$g := -\tau \cdot (u \cdot \nabla) u \,|\, B^c_T,$$
$$h^{(l)} := (\tau \cdot u_l \cdot u_i \,|\, B^c_T)_{1 \leq i \leq 3}$$

for $1 \leq l \leq 3$. By [6, Lemma IX.7.1, (IX.7.9)], we have $g, h^{(l)} \in L^{6/5}(B^c_T)^3$.

Let a fundamental solution $(E_{jk})_{1 \leq j \leq 4,\, 1 \leq k \leq 3}$ of the Oseen system be defined as in [3, Definition 4.1], for example. Here we will only need the following properties of this solution ([7, (1.39)]):

$$|E_{jk}(z)| \leq \mathcal{C} \cdot \nu^{-1}_{-1}(z,\tau), \quad (8)$$
$$|D_m E_{jk}(z)| \leq \mathcal{C} \cdot \varrho(z,\tau),$$
$$|D_n D_m (E_{jk} - U_{jk})(z)| \leq \mathcal{C} \cdot \tau \cdot \nu^{-2}_{-1}(z,\tau)$$

for $z \in \mathbb{R}^3 \backslash \{0\}$, $j, k, m, n \in \{1, 2, 3\}$, where U_{jk} denotes the velocity part of the Stokes fundamental solution, that is,

$$U_{jk}(z) := (8 \cdot \pi)^{-1} \cdot \left(\delta_{jk} \cdot |z|^{-1} + z_j \cdot z_k \cdot |z|^{-3} \right)$$

for $j, k \in \{1, 2, 3\}$, $z \in \mathbb{R}^3 \backslash \{0\}$. If $V \subset \mathbb{R}^3$ is measurable and $\varphi \in L^{6/5}(V)^3$, define $\mathcal{R}(\varphi) : \mathbb{R}^3 \mapsto \mathbb{R}^3$, $\mathcal{S}(\varphi) : \mathbb{R}^3 \mapsto \mathbb{R}$ by

$$\mathcal{R}_j(\varphi)(x) := \int_V \sum_{k=1}^{3} E_{jk}(x-y) \cdot \varphi_k(y) \, dy,$$
$$\mathcal{S}(\varphi)(x) := \int_V \sum_{k=1}^{3} E_{4k}(x-y) \cdot \varphi_k(y) \, dy$$

for $j \in \{1, 2, 3\}$ and a.e. $x \in \mathbb{R}^3$ ("volume potentials"). For V, φ as before, we have $\mathcal{R}(f) \in W^{1,1}_{loc}(\mathbb{R}^3)^3$, and the integral

$$\int_V \sum_{k=1}^{3} D_l E_{jk}(x-y)\, \varphi_k(y)\, dy$$

exists and equals $D_l \mathcal{R}_j(\varphi)(x)$ for a. e. $x \in \mathbb{R}^3$, $1 \le j, l \le 3$. If in addition $\varphi \in L^\infty_{loc}(V)^3$, then $\mathcal{R}(f) \in C^1(\mathbb{R}^3)^3$, and the qualification "a. e." may be dropped.

Referring to [3, (5.23), (5.26) - (5.29)] with Ω, S, S_1 replaced by B_T, $2 \cdot T$, $3 \cdot T$, and to [3, Theorem 4.9] with S replaced by T, we find for $x \in B^c_{3 \cdot T}$:

$$\begin{aligned} |u(x)| &\le \mathcal{C} \cdot \nu^{-1}_{-1}(x, \tau) + |\mathcal{R}(g)(x)|, \\ |\nabla u(x)| &\le \mathcal{C} \cdot \varrho(x, \tau) + |\nabla \mathcal{R}(g)(x)|, \\ |\pi(x)| &\le \mathcal{C} \cdot |x|^{-2} + |\mathcal{S}(g)(x)|. \end{aligned} \tag{9}$$

Since

$$\int_{B^c_R} |f \cdot u|\, dx \le \mathcal{C} \cdot R^{-\sigma+9/4} \cdot \|u \,|\, B^c_T\|_3$$

($R \in (T, \infty)$) by (2), and $\|u \,|\, B^c_T\|_3 < \infty$ by (7), we may argue as in [6, p. 131], to obtain $|\mathcal{R}(g)(x)| \le \mathcal{C} \cdot |x|^{-63/64}$ for $x \in B^c_{2 \cdot T}$, hence with (9) and (6): $|u(x)| \le \mathcal{C} \cdot |x|^{-63/64}$ $(x \in B^c_T)$. This estimate and the relation $\|\nabla u\|_2 < \infty$ (see (3)) allow us to transform $\mathcal{R}(f)(x)$ by a partial integration. We obtain for $x \in \overline{B_T}^c$, $1 \le j \le 3$:

$$\mathcal{R}_j(g)(x) = -\sum_{l=1}^{3} D_l \mathcal{R}_j(h^{(l)})(x) + \int_{\partial B_T} \sum_{k,l=1}^{3} E_{jk}(x-y) \cdot h^{(l)}_k(y) \cdot y_l / T\, do_y. \tag{10}$$

Using the estimate $|u(x)| \le \mathcal{C} \cdot |x|^{-63/64}$ $(x \in B^c_T)$ again, as well as (8) and (7), we find as in the proof of [6, Theorem IX.8.1] that $|\, D_l \mathcal{R}_j(h^{(l)})(x)\,| \le \mathcal{C} \cdot |x|^{-47/32}$ for $x \in B_{2 \cdot T}$, $1 \le j, l \le 3$. The boundary integral on the right-hand side of (10) may easily be estimated by $\mathcal{C} \cdot (\nu^{-1}_{-1}(x, \tau) + |x|^{-3/2})$ $(x \in B^c_{2 \cdot T})$. (Add and subtract the term $\sum_{k,l=1}^{3} E_{jk}(x) \cdot h^{(l)}_k(y) \cdot y_l / T$ in this integral, and then use the mean-value theorem, (8) and (6).) Thus we may conclude from (6), (10) and (9):

$$|u(x)| \;\le\; \mathcal{C} \cdot (\nu^{-1}_{-1}(x, \tau) + |x|^{-47/32}) \tag{11}$$

for $x \in B^c_T$. Next we begin estimating the term $|\nabla u(x)|$. To this end, take $j, l \in \{1, 2, 3\}$. We split $D_l \mathcal{R}(g)(x)$ into four parts: $D_l \mathcal{R}(g)(x) = \sum_{m=1}^{4} I_m(x)$, with

$$I_1(x) := -\int_{B^c_T \setminus B_T(x)} \sum_{i,k=1}^{3} D_i D_l E_{jk}(x-y) \cdot h^{(k)}_i(y)\, dy,$$

$$I_2(x) := \int_{B_T(x)} \sum_{k=1}^{3} D_l E_{jk}(x-y) \cdot g_k(y)\, dy,$$

$$I_3(x) := -\int_{\partial B_T(x)} \sum_{i,k=1}^{3} D_l E_{jk}(x-y) \cdot h^{(k)}_i(y) \cdot (x-y)_i / T\, dy,$$

$$I_4(x) := -\int_{\partial B_T} \sum_{i,k=1}^{3} D_l E_{jk}(x-y) \cdot h^{(k)}_i(y) \cdot (-y_i / T)\, dy, \tag{12}$$

for $x \in B_{2 \cdot T}$. By adding and subtracting the term $\sum_{i,k=1}^{3} D_l E_{jk}(x) \cdot h^{(k)}_i(y) \cdot (-y_i / T)$ in the integral defining $I_4(x)$, we get with the mean-value theorem, (8) and (6):

$$|I_4(x)| \le \mathcal{C} \cdot \varrho(x, \tau) \;\text{ for }\; x \in B^c_{2 \cdot T}. \tag{13}$$

In addition, $|I_3(x)| \le \mathcal{C} \cdot |x|^{-2}$ $(x \in B^c_{2 \cdot T})$, as follows from (8) and (11). Arguing as in the proof of [6, (IX.8.30), (IX.8.31)], we obtain with (8), (4), (11): $|I_1(x)| \le \mathcal{C} \cdot |x|^{-31/16}$ for $x \in B^c_{2 \cdot T}$. Thus, in view of (9) and the previous decomposition of $\mathcal{R}(g)(x)$, we may conclude

$$|D_l u_j(x)| \;\le\; \mathcal{C} \cdot \big(\, \varrho(x, \tau) + |x|^{-31/16}\,\big) + |I_2(x)| \;\; (x \in B^c_{2 \cdot T}). \tag{14}$$

For a first estimate of $I_2(x)$, we observe that $|\nabla u(x)| \le \mathcal{C}$ $(x \in B^c_T)$ by (6), (7). Thus we get by (11) and (8): $|I_2(x)| \le \mathcal{C} \cdot |x|^{-1}$ $(x \in B^c_{2 \cdot T})$. It follows with (14) and (6) that $|D_l u_j(x)| \le \mathcal{C} \cdot |x|^{-1}$ $(x \in B^c_T)$. This result, (11) and (8) yield $|I_2(x)| \le \mathcal{C} \cdot |x|^{-2}$ $(x \in B_{2 \cdot T})$, hence with (14) and (6):

$$|D_l u_j(x)| \;\le\; \mathcal{C} \cdot \big(\, \varrho(x, \tau) + |x|^{-31/16}\,\big) \tag{15}$$

for $x \in B_T^c$, $1 \le j,l \le 3$. Inequalities (11) and (15) imply

$$|g(y)| \;\le\; \mathcal{C}\cdot\big(\,\nu_{-1}^{-5/2}(y,\tau)+|y|^{-7/2+3/16}\,\big) \quad (16)$$

for $y \in B_T^c$. By (5), this means in particular that $|g(x)|$ is bounded by $\mathcal{C}\cdot\nu_{-1}^{-5/2+3/16}(x,\tau)$ for $x \in B_T^c$, hence by [7, Theorem 3.1]: $|\mathcal{R}(g)(x)| \le \mathcal{C}\cdot\nu_{-1}^{-1}(x,\tau)$ for $x \in \mathbb{R}^3\backslash\{0\}$. Thus we get by (9):

$$|u(x)| \;\le\; \mathcal{C}\cdot\nu_{-1}^{-1}(x,\tau) \;\; (x \in B_T^c). \qquad (17)$$

This means that we have shown the result on u stated in Theorem 1. Our next aim is to estimate $|\nabla u(x)|$ by $\mathcal{C}\cdot\varrho(x,\tau)$. To this end, let $j,l \in \{1,\,2,\,3\}$ and $x \in B_{2\cdot T}^c$. Then we have $D_l\mathcal{R}_j(g)(x) = \sum_{m=1}^3 J_m(x)$, with

$$J_m(x) := \int_{U_m}\sum_{k=1}^{3} D_lE_{jk}(x-y)\cdot g(y)\;dy$$

for $m \in \{1,\,2,\,3\}$, where $U_1 := B_{2\cdot|x|}^c$, $U_2 := B_{2\cdot|x|}\backslash B_{|x|/2}$, $U_3 := B_{|x|/2}\backslash B_T$. It may be deduced from (8), (16) and (4) that $|J_1(x)| \le \mathcal{C}\cdot|x|^{-2}$. Moreover, [7, Theorem 3.2] yields

$$\int_{\mathbb{R}^3}\sum_{k=1}^{3}|D_lE_{jk}(x-y)|\cdot\nu_{-1}^{-5/2}(y,\tau)\;dy \;\le\; \mathcal{C}\cdot\nu_{-3/2}^{-3/2}(x,\tau).$$

By (8), $\int_{U_2}|D_lE_{jk}(x-y)|\;dx \le \mathcal{C}\cdot|x|$, hence $\int_{U_2}|D_lE_{jk}(x-y)|\cdot|y|^{-7/2+3/16}\;dx \le \mathcal{C}\cdot|x|^{-2}$. In view of (16), we thus have found $|J_2(x)| \le \varrho(x,\tau)$. Turning to $J_3(x)$, we first get by a partial integration that $J_3(x) = \sum_{m=1}^3 H_m(x) + I_4(x)$, with

$$H_1(x) := -\int_{B_{|x|/2}\backslash B_T}\sum_{i,k=1}^{3} D_iD_l(E_{jk}-U_{jk})(x-y)\cdot h_i^{(k)}(y)\;dy,$$

$$H_3(x) := -\int_{\partial B_{|x|/2}}\sum_{i,k=1}^{3} D_lE_{jk}(x-y)\cdot h_i^{(k)}(y)\cdot 2\cdot y_i/|x|\;dy.$$

The term $H_2(x)$ is defined as $H_1(x)$, but with the kernel $E_{jk}-U_{jk}$ replaced by U_{jk}. For the definition of $I_4(x)$, see (12). Noting that by (17),

$$|h_i^{(k)}(y)| \le \nu_{-2}^{-2}(y,\tau)\;(y \in B_T^c,\; 1 \le i,k \le 3), \quad (18)$$

we get with [7, Theorem 3.3] that

$$|H_1(x)| \;\le\; \mathcal{C}\cdot\nu_{-1}^{-2}(x,\tau)\cdot\big(1+|\ln(\tau\cdot|x|)|\big).$$

A simple calculation which makes use of the estimate $|h_i^{(k)}(y)| \le \mathcal{C}\cdot|y|^{-2}$ $(y \in B_T^c)$ yields $|H_2(x)| \le \mathcal{C}\cdot|x|^{-2}$. Observing that $\varrho(x-y,\tau) \le \mathcal{C}\cdot|x|^{-3/2}$ for $y \in \partial B_{|x|/2}$, and recalling (18), (8) and (4), we obtain $|H_3(x)| \le \mathcal{C}|x|^{-5/2}$. Combining these estimates of $H_1(x)$ - $H_3(x)$, the estimate of $I_4(x)$ in (13), and the inequalities we found for $|J_1(x)|$ and $|J_2(x)|$, we obtain

$$|\,D_l\mathcal{R}_j(g)(x)\,| \;\le\; \mathcal{C}\cdot\big(\,\varrho(x,\tau) + \nu_{-1}^{-2}(x,\tau)\cdot\big(1+|\ln(\tau\cdot|x|)|\,\big)\,\big).$$

Now we may conclude with (9) and (6) that for $x \in B_T^c$, $1 \le j,l \le 3$,

$$|D_lu_j(x)| \;\le\; \mathcal{C}\cdot\big(\,\varrho(x,\tau) + \nu_{-1}^{-2}(x,\tau)\cdot\big(1+|\ln(\tau\cdot|x|)|\,\big)\,\big).$$

It follows with inequality (17) that $|g(y)| \le \mathcal{C}\cdot\nu_{-1}^{-5/2}(y,\tau)$ $(y \in B_T^c)$. Now [7, Theorem 3.2] yields

$$|\nabla\mathcal{R}(g)(x)| \le \mathcal{C}\cdot\nu_{-3/2}^{-3/2}(x,\tau)\;\;(x \in \mathbb{R}^3\backslash\{0\}),$$

hence by (9): $|\nabla u(x)| \le \mathcal{C}\cdot\varrho(x,\tau)$ for $x \in B_T^c$. This leaves us to evaluate $|\pi(x)|$. To this end, we observe that our preceding estimate of $g(y)$ implies in particular $|g(y)| \le \mathcal{C}\cdot\nu_{-3/4}^{-5/2}(y,\tau)$ $(y \in B_T^c)$. Therefore [7, Theorem 3.4] yields that $|\mathcal{S}(g)(x)| \le \mathcal{C}\cdot|x|^{-2}$ for $x \in \mathbb{R}^3\backslash\{0\}$, hence with (9): $|\pi(x)| \le \mathcal{C}\cdot|x|^{-2}$ $(x \in B_T^c)$.

This completes the proof of Theorem 1.

3 Conclusion

We studied the asymptotic behaviour of 3D exterior stationary incompressible Navier-Stokes flows. It was shown that that the velocity, the gradient of the velocity, and the pressuere each exhibit a specific decay behaviour, which in the case of the velocity and its gradient is inhomogeneous (wake phenomenon).

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