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Shape Optimization in Problems Governed by Generalised Navier–Stokes Equations: Existence Analysis

JAROSLAV HASLINGER[1,3], JOSEF MÁLEK[2,4], JAN STEBEL[1,5]

[1] Department of Numerical Mathematics,
Faculty of Mathematics and Physics, Charles University,
Sokolovská 83, CZ-186 75 Praha 8, CZECH REPUBLIC

[2] Mathematical Institute, Charles University,
Sokolovská 83, CZ-186 75 Praha 8, CZECH REPUBLIC

[3] hasling@karlin.mff.cuni.cz, [4] malek@karlin.mff.cuni.cz, [5] stebel@karlin.mff.cuni.cz

Abstract: We study a shape optimization problem for a paper machine headbox which distributes a mixture of water and wood fibers in the paper manufacturing process. The aim is to find a shape which a priori ensures the given velocity profile on the outlet part. The state problem is represented by the generalised Navier-Stokes system with nontrivial boundary conditions. The objective of this paper is to prove the existence of an optimal shape.

Key Words: Optimal shape design, paper machine headbox, incompressible non-Newtonian fluid, algebraic turbulence model.

1 Introduction

This contribution deals with shape optimization of a paper machine headbox. The headbox shape and the fluid flow phenomena taking place there largely determines the quality of the produced paper. The first flow passage in the headbox is a dividing manifold, called the header. It is designed to distribute the fiber suspension on the wire so that the produced paper has an optimal basis weight and fiber orientation across the whole width of a paper machine. The aim of this work is to find an optimal shape for the back wall of the header so that the outlet flow rate distribution from the headbox results in an optimal paper quality.

This work was motivated by some previous papers: The fluid flow model which is used here has been derived and studied numerically in Hämäläinen [2]. The shape optimization problem has also been solved numerically and the results are presented in Hämäläinen, Mäkinen and Tarvainen [1], see also Haslinger and Mäkinen [3]. Both fluid flow model and shape optimization problem have been studied there formally without establishing existence results. Therefore our goal is to give the theoretical analysis of the flow equations and of the whole optimization problem.

We assume a steady flow of an incompressible liquid with an algebraic turbulence model, which is very similar to model of non-Newtonian fluids with shear dependent viscosity (see Rajagopal [11], Málek, Nečas, Rokyta, Růžička [8] and Málek, Rajagopal, and Růžička [9] for more details about non-Newtonian fluids).

The text is organized as follows. In Section 2 we present the fluid flow model and analyze the existence of a solution. The existence proof is based on appropriate energy estimates and the Galerkin method. A shape optimization problem is formulated in Section 3 and the existence of an optimal shape is established. The continuous dependence of solutions to state problems with respect to shape variations is the most important result of this part.

The detailed mathematical analysis together

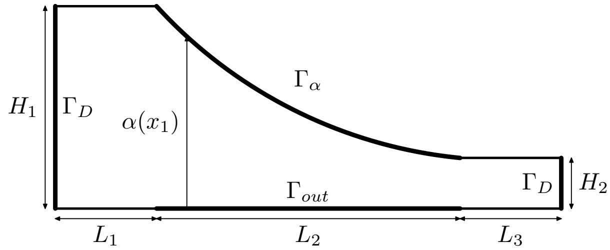

Figure 1: Geometry of $\Omega(\alpha)$ and parts of the boundary $\partial\Omega(\alpha)$.

with a justification of the fluid model can be found in [4].

2 Steady flow of a non-Newtonian fluid

For describing the fluid flow in the header we shall use a two–dimensional stationary model. First we define the geometry of the problem.

2.1 Description of admissible domains

Let $L_1, L_2, L_3 > 0$, $H_1 \geq H_2 > 0$, $\alpha_{max} \geq \alpha_{min} > 0$, $\gamma > 0$ be given and suppose that $\alpha \in \mathcal{U}_{ad}$, where

$$\mathcal{U}_{ad} = \Big\{\alpha \in C^{0,1}([0,L]);\ \alpha_{min} \leq \alpha \leq \alpha_{max},\ \alpha|_{[0,L_1]} = H_1,\ \alpha|_{[L_1+L_2,L]} = H_2,\ |\alpha'| \leq \gamma \text{ a.e. in } [0,L]\Big\}. \quad (1)$$

Here $C^{0,1}([0,L])$ denotes the set of Lipschitz continuous functions on $[0,L]$ and $L = L_1+L_2+L_3$. With any $\alpha \in \mathcal{U}_{ad}$ we associate the domain $\Omega(\alpha)$, see Fig. 1:

$$\Omega(\alpha) = \Big\{(x_1,x_2) \in \mathbb{R}^2; 0 < x_1 < L,\ 0 < x_2 < \alpha(x_1)\Big\} \quad (2)$$

and introduce the system of admissible domains

$$\mathcal{O} = \big\{\Omega;\ \exists\ \alpha \in \mathcal{U}_{ad} :\ \Omega = \Omega(\alpha)\big\}.$$

Further we shall need the domains

$$\widehat{\Omega} = (0,L) \times (0,\alpha_{max})$$

and

$$\Omega_0 = \big((0,L_1)\times(0,H_1)\big) \cup \big((0,L)\times(0,\alpha_{min})\big) \cup \big((L_1+L_2,L)\times(0,H_2)\big).$$

Clearly $\Omega(\alpha) \in C^{0,1}$ for all $\alpha \in \mathcal{U}_{ad}$, where $C^{0,1}$ is the system of bounded domains with Lipschitz continuous boundaries. We shall denote the parts of the boundary $\partial\Omega(\alpha)$ as follows (see Fig. 1):

$$\begin{aligned}
\Gamma_D &= \Big\{(x_1,x_2) \in \partial\Omega(\alpha); x_1 = 0 \text{ or } x_1 = L\Big\},\\
\Gamma_{out} &= \Big\{(x_1,x_2) \in \partial\Omega(\alpha); L_1 \leq x_1 \leq L_1+L_2,\ x_2 = 0\Big\},\\
\Gamma_\alpha &= \Big\{(x_1,x_2) \in \partial\Omega(\alpha); L_1 \leq x_1 \leq L_1+L_2,\ x_2 = \alpha(x_1)\Big\},\\
\Gamma_f &= \partial\Omega(\alpha) \setminus \big(\Gamma_D \cup \Gamma_{out} \cup \Gamma_\alpha\big).
\end{aligned}$$

The components Γ_D, Γ_{out} and Γ_f are fixed for every $\alpha \in \mathcal{U}_{ad}$.

2.2 Classical formulation of the state problem

The fluid motion in $\Omega(\alpha)$ is described by the generalised Navier–Stokes system

$$\left.\begin{aligned} -\mathrm{div}\ T(p,D(u)) + \rho\,\mathrm{div}(u\otimes u) &= 0\\ \mathrm{div}\ u &= 0 \end{aligned}\right\} \text{ in } \Omega(\alpha). \quad (3)$$

Here u means the velocity, p the pressure, ρ is the density of the fluid and the stress tensor T is defined by the following formulae:

$$T_{ij}(p,D(u)) = -p\delta_{ij} + 2\mu(|D(u)|)D_{ij}(u),\quad i,j = 1,2,$$

$$\mu(|D(u)|) := \mu_0 + \mu_t(|D(u)|) = \mu_0 + \rho l_{m,\alpha}^2|D(u)|,$$

where $\mu_0 > 0$ is a constant laminar viscosity and $\mu_t(|D(u)|)$ stands for a turbulent viscosity. The function $l_{m,\alpha}$ represents a mixing length in the algebraic model of turbulence and it has the following form (see Hämäläinen, Mäkinen and Tarvainen [1] for more details):

$$l_{m,\alpha}(x) = \frac{1}{2}\alpha(x_1)\left[0.14 - 0.08\left(1 - \frac{2d_\alpha(x)}{\alpha(x_1)}\right)^2 - 0.06\left(1 - \frac{2d_\alpha(x)}{\alpha(x_1)}\right)^4\right],$$

where $d_\alpha(x) = \min\{x_2, \alpha(x_1) - x_2\}, x \in \Omega(\alpha)$.

The equations are completed by the following boundary conditions:

$$\begin{array}{rcll} u &=& 0 & \text{on } \Gamma_f \cup \Gamma_\alpha, \\ u &=& u_D & \text{on } \Gamma_D, \\ u\cdot\tau = u_1 &=& 0 & \text{on } \Gamma_{out}, \\ T_{22} &=& -\sigma|u_2|u_2 & \text{on } \Gamma_{out}, \end{array} \tag{4}$$

where ν, τ stands for the unit normal, tangential vector, respectively and $\sigma > 0$ is a given suction coefficient. The condition $(4)_4$ originates in the homogenization of a complex geometry of Γ_{out} (for more details we refer to Hämäläinen [2]).

2.3 Weak formulation of the state problem

Throughout the paper we assume that there exists a function $u_0 \in \left(W^{1,3}(\Omega_0)\right)^2$, which satisfies the Dirichlet boundary conditions in the sense of traces, i.e.

$$u_0|_{\Gamma_D} = u_D,\ \ u_0|_{\partial\Omega_0\setminus(\Gamma_D\cup\Gamma_{out})} = 0,\ \ u_0\cdot\tau|_{\Gamma_{out}} = 0$$

and, in addition, div $u_0 = 0$ in Ω_0. We extend u_0 by zero on $\widehat{\Omega}\setminus\Omega_0$. Then, due to the boundary conditions, $u_0 \in \left(W^{1,3}(\widehat{\Omega})\right)^2$ and div $u_0 = 0$ in $\widehat{\Omega}$.

2.3.1 Function spaces

For any $\alpha \in \mathcal{U}_{ad}$ we introduce the following function spaces:

$$\mathcal{V}(\alpha) := \left\{\varphi \in (\mathcal{C}^\infty(\overline{\Omega(\alpha)}))^2;\ \text{div}\ \varphi = 0 \text{ in } \Omega(\alpha)\right\},$$

$$\mathcal{V}_0(\alpha) := \Big\{\varphi = (\varphi_1,\varphi_2) \in \mathcal{V}(\alpha);\ \ \varphi_1 \in \mathcal{C}_0^\infty(\Omega(\alpha)),$$
$$\text{dist}(\text{supp}(\varphi_2), \partial\Omega(\alpha) \setminus \Gamma_{out}) > 0\Big\},$$

$$\begin{array}{rcl} W(\alpha) &:=& \overline{\mathcal{V}(\alpha)}^{\|\cdot\|_\alpha}, \\ W_0(\alpha) &:=& \overline{\mathcal{V}_0(\alpha)}^{\|\cdot\|_\alpha}, \\ W_{u_0}(\alpha) &:=& \{v \in W(\alpha);\ v - u_0 \in W_0(\alpha)\}, \end{array}$$

where the norm $\|\cdot\|_\alpha$ is defined by

$$\|v\|_\alpha := \|v\|_{1,2,\Omega(\alpha)} + \|M_\alpha|D(v)|\|_{3,\Omega(\alpha)},$$

with

$$M_\alpha(x) := \Big(l_{m,\alpha}(x)\Big)^{2/3}, x \in \Omega(\alpha). \tag{5}$$

Here we use standard notations: the norm in $L^s(\Omega(\alpha))$, $W^{k,s}(\Omega(\alpha))$ will be denoted by $\|\ \|_{s,\Omega(\alpha)}$, $\|\ \|_{k,s,\Omega(\alpha)}$, respectively, in what follows. We shall also use the Einstein summation convention, i.e. $a_ib_i := \sum_{i=1}^n a_ib_i$.

Lemma 2.1. *$W(\alpha)$ and $W_0(\alpha)$ are separable reflexive Banach spaces.*

Definition 2.1. *Define the operator $A_\alpha : W(\alpha) \to \left(W(\alpha)\right)^*$ by the formula*

$$\left\langle A_\alpha(v), w\right\rangle_\alpha := \int_{\Omega(\alpha)} M_\alpha^3|D(v)|D_{ij}(v)D_{ij}(w)dx;$$
$$v, w \in W(\alpha).$$

The symbol $\langle\cdot,\cdot\rangle_\alpha$ denotes the duality pairing between $\left(W(\alpha)\right)^*$ and $W(\alpha)$.

Remark 2.1. *Since $M_\alpha = 0$ on $\partial\Omega(\alpha)\setminus\Gamma_D$, it can be extended by zero on $\widehat{\Omega}\setminus\Omega(\alpha)$. The resulting function, which is continuous in $\overline{\widehat{\Omega}}$ and which will be used in the subsequent analysis, will be denoted by $\tilde{M}_\alpha$.*

Lemma 2.2. *(Some properties of M_α and A_α, $\alpha \in \mathcal{U}_{ad}$)*

(i) If $\alpha_n \rightrightarrows \alpha$ in $[0, L]$ then $\tilde{M}_{\alpha_n} \rightrightarrows \tilde{M}_\alpha$ in $\overline{\widehat{\Omega}}$.

(ii) A_α is monotone in $W(\alpha)$:

$$\left\langle A_\alpha(v) - A_\alpha(w), v - w\right\rangle_\alpha \geq 0\ \forall v, w \in W(\alpha),$$

and strictly monotone in $W_0(\alpha)$, i.e. the previous inequality is sharp for $v \neq w$, where $v, w \in W_0(\alpha)$.

(iii) A_α is continuous in $W(\alpha)$.

Definition 2.2. *For every $u, v, w \in \left(W^{1,2}(\Omega(\alpha))\right)^2$ we define the trilinear form b_α:*

$$b_\alpha(u, v, w) := \int_{\Omega(\alpha)} u_j\frac{\partial v_i}{\partial x_j}w_i\ dx.$$

Remark 2.2. *The same analysis can be done for any weight function $M_\alpha : \overline{\widehat{\Omega}} \mapsto \mathbb{R}$ satisfying the following conditions:*

(i) $\forall\alpha \in \mathcal{U}_{ad}\ \ M_\alpha \in \mathcal{C}(\overline{\widehat{\Omega}})$;

(ii) $\forall\alpha \in \mathcal{U}_{ad}$ it holds that $M_\alpha|_{\Omega(\alpha)} > 0$;

(iii) $\forall\alpha_n, \alpha \in \mathcal{U}_{ad}\ \ \alpha_n \rightrightarrows \alpha$ in $[0, L] \Rightarrow M_{\alpha_n} \rightrightarrows M_\alpha$ in $\overline{\widehat{\Omega}}$.

2.3.2 Definition of a weak solution

We are now ready to give the weak formulation of the state problem. It can be formally derived by multiplying the equations (3) by a smooth solenoidal test function φ and integrating over $\Omega(\alpha)$ with the use of the Green theorem.

Definition 2.3. *A function $u := u(\alpha)$ is said to be a weak solution of the state problem $(\mathcal{P}(\alpha))$ iff*

- *$u \in W_{u_0}(\alpha)$,*
- *for every $\varphi \in W_0(\alpha)$ it holds:*

$$2\mu_0 \int_{\Omega(\alpha)} D_{ij}(u) D_{ij}(\varphi)\, dx + 2\rho \langle A_\alpha(u), \varphi \rangle_\alpha + \rho b_\alpha(u, u, \varphi) + \sigma \int_{\Gamma_{out}} |u_2| u_2 \varphi_2 \, dS = 0. \quad (6)$$

Remark 2.3. *Since $\varphi = 0$ on $\partial\Omega(\alpha) \setminus \Gamma_{out}$ and div $\varphi = 0$ in $\Omega(\alpha)$, the pressure disappeared from the weak formulation.*

Next the existence of a weak solution to $(\mathcal{P}(\alpha))$ on a fixed domain $\Omega(\alpha)$, $\alpha \in \mathcal{U}_{ad}$ will be examined. Thus for simplicity the letter α in the argument will be omitted (we shall write $\Omega := \Omega(\alpha)$, $W := W(\alpha)$, $b := b_\alpha$ etc. in what follows).

2.4 Energy estimates

Recall that the function u_0 is now defined in the whole $\widehat{\Omega}$ and it does not depend on $\alpha \in \mathcal{U}_{ad}$.

Theorem 2.1. *Let $\|\nabla u_0\|_{3,\widehat{\Omega}}$ be sufficiently small and $\sigma > \frac{\rho}{2}$. Then there exists a constant $C_E := C_E(\|\nabla u_0\|_{3,\widehat{\Omega}})$ such that for any weak solution u of $(\mathcal{P}(\alpha))$ the following estimate holds:*

$$\|\nabla u\|^2_{2,\Omega} + \|M|D(u)|\|^3_{3,\Omega} + \int_{\Gamma_{out}} |u_2|^3 \, dS \le C_E. \quad (7)$$

Remark 2.4. *From the proof it follows that estimate (7) holds with a constant C_E independent of $\alpha \in \mathcal{U}_{ad}$.*

Remark 2.5. *Let us comment on the assumptions of Theorem 2.1.*

(i) The condition $\sigma > \frac{\rho}{2}$ can be possibly satisfied by adjusting the outflow properties of the headbox.

(ii) Assume that there exists a constant $\overline{C} > 0$ such that

$$\forall \alpha \in \mathcal{U}_{ad} \quad \|M_\alpha^{-1}\|_{2,\Omega(\alpha)} \le \overline{C}. \quad (8)$$

Then Theorem 2.1 holds for any $\|\nabla u_0\|_{3,\widehat{\Omega}}$ with a constant $C'_E > 0$ independent of α, provided that $\sigma > \frac{\rho}{2}$.

Remark 2.6. *A direct calculation shows that the function M_α defined in (5) does not satisfy (8) since $M_\alpha \approx x_2^{2/3}$ in a vicinity of $\partial\Omega(\alpha) \setminus \overline{\Gamma_D}$. This condition will be satisfied if M_α decays as $x_2^{1/2-\epsilon}$ with $\epsilon > 0$ arbitrarily small.*

2.5 Existence and uniqueness

The existence proof is based on the Galerkin method. It is easy to show that the Galerkin approximation exists on any finite dimensional subspace of $W_0(\alpha)$. Using the energy estimate (7) we obtain the following existence result.

Theorem 2.2 (Existence of a weak solution). *Let the assumptions of Theorem 2.1 be satisfied. Then there exists a weak solution of $(\mathcal{P}(\alpha))$.*

Theorem 2.3 (Uniqueness). *Let all the assumptions of Theorem 2.1 be satisfied and $\|\nabla u_0\|_{3,\widehat{\Omega}}$ be small enough. Then there exists a unique solution to $(\mathcal{P}(\alpha))$.*

Remark 2.7. *Let us observe that the bound guaranteeing uniqueness of the solution to $(\mathcal{P}(\alpha))$ is also independent of $\alpha \in \mathcal{U}_{ad}$.*

3 Shape optimization problem

The aim of this part is to formulate a shape optimization problem and to prove the existence of its solution.

3.1 Formulation of the problem

We proved that, under certain assumptions, which do not depend on a particular choice of $\Omega(\alpha) \in \mathcal{O}$, there exists at least one weak solution of the state problem $(\mathcal{P}(\alpha))$. Let $\mathcal{G}$ be the graph of the control–to–state (generally multi-valued) mapping:

$$\mathcal{G} := \{(\alpha, u);\ \alpha \in \mathcal{U}_{ad},\ u \text{ is a weak solution of } (\mathcal{P}(\alpha))\}.$$

Further, let us define the cost functional $J : \mathcal{G} \to \mathbb{R}$ by

$$J : (\alpha, u) \mapsto \int_{\tilde{\Gamma}} |u_2 - z_D|^2 \, dS, \; u = (u_1, u_2), \quad (9)$$

where $z_D \in L^2(\tilde{\Gamma})$ is a given function representing the desired outlet velocity profile and $\tilde{\Gamma} \subset \Gamma_{out}$. This choice of J reflects the optimization goal formulated in Section 1.

We now formulate the following problem:

$$\text{Find } (\alpha^*, u^*) \in \mathcal{G} \text{ such that}$$
$$J(\alpha^*, u^*) \le J(\alpha, u) \; \forall (\alpha, u) \in \mathcal{G}. \qquad (\mathbb{P})$$

Next we introduce convergence of a sequence of domains.

Definition 3.1. *Let $\{\Omega(\alpha_n)\}$, $\alpha_n \in \mathcal{U}_{ad}$ be a sequence of domains. We say that $\{\Omega(\alpha_n)\}$ converges to $\Omega(\alpha)$, shortly $\Omega(\alpha_n) \rightsquigarrow \Omega(\alpha)$, iff $\alpha_n \rightrightarrows \alpha$ in $[0, L]$.*

As a direct consequence of the Arzelà–Ascoli theorem we have the following compactness result.

Lemma 3.1. *System $\mathcal{O}$ is compact with respect to convergence introduced in Definition* 3.1.

3.2 Existence of an optimal shape

First let us recall that the function u_0 which realizes the boundary conditions is the same for all domains $\Omega \in \mathcal{O}$. We now rewrite $(\mathcal{P}(\alpha))$, $\alpha \in \mathcal{U}_{ad}$ using the formulation on the fixed domain $\widehat{\Omega}$:

$$2\mu_0 \int_{\widehat{\Omega}} D_{ij}(\tilde{u}(\alpha)) D_{ij}(\tilde{\varphi}) \, dx + 2\rho \langle \tilde{A}_\alpha(\tilde{u}(\alpha)), \tilde{\varphi} \rangle_{\widehat{\Omega}}$$
$$+ \rho b_{\widehat{\Omega}}(\tilde{u}(\alpha), \tilde{u}(\alpha), \tilde{\varphi}) + \sigma \int_{\Gamma_{out}} |\tilde{u}_2(\alpha)| \tilde{u}_2(\alpha) \tilde{\varphi}_2 \, dS = 0$$
$$\forall \varphi \in W_0(\alpha), \quad (\widehat{\mathcal{P}}(\alpha))$$

where the symbol $\tilde{}$ stands for the zero extension of functions from $\Omega(\alpha)$ on $\widehat{\Omega}$,

$$\langle \tilde{A}_\alpha(\tilde{u}(\alpha)), \tilde{\varphi} \rangle_{\widehat{\Omega}} := \int_{\widehat{\Omega}} \tilde{M}_\alpha^3 |D(\tilde{u}(\alpha))| D_{ij}(\tilde{u}(\alpha)) D_{ij}(\tilde{\varphi}) \, dx,$$

$$b_{\widehat{\Omega}}(\tilde{u}(\alpha), \tilde{u}(\alpha), \tilde{\varphi}) := \int_{\widehat{\Omega}} \tilde{u}_j(\alpha) \frac{\partial \tilde{u}_i(\alpha)}{\partial x_j} \tilde{\varphi}_i \, dx.$$

Further let

$$\widehat{W}(\alpha) := \Big\{ v \in \big(W^{1,2}(\Omega(\alpha))\big)^2; \; \operatorname{div} v = 0 \text{ in } \Omega(\alpha), \; M_\alpha |D(v)| \in L^3(\Omega(\alpha)) \Big\}$$

and define

$$\widehat{W}_{u_0}(\alpha) := \Big\{ v \in \widehat{W}(\alpha); \; v \text{ satisfies the Dirichlet conditions } (4)_1 - (4)_3 \text{ on } \partial\Omega(\alpha) \Big\}.$$

Remark 3.1. *It holds that $W_{u_0}(\alpha) \subseteq \widehat{W}_{u_0}(\alpha)$. The question arises, if these spaces are identical. This is in fact the density problem which remains still open.*

Theorem 2.1 gives the following energy estimate:

$$\|\nabla \tilde{u}(\alpha)\|^2_{2,\widehat{\Omega}} + \|\tilde{M}_\alpha |D(\tilde{u}(\alpha))|\|^3_{3,\widehat{\Omega}} + \int_{\Gamma_{out}} |u_2(\alpha)|^3 \, dS \le C_E(\|\nabla u_0\|_{3,\widehat{\Omega}}) \quad (11)$$

for every $(\alpha, u(\alpha)) \in \mathcal{G}$ with the constant $C_E(\|\nabla u_0\|_{3,\widehat{\Omega}})$ independent of α provided that assumptions of Theorem 2.1 are satisfied.

Theorem 3.1. *Let $\alpha_n \rightrightarrows \alpha$ in $[0, L]$, $\alpha_n, \alpha \in \mathcal{U}_{ad}$ and $u_n := u(\alpha_n)$ be a solution of $(\mathcal{P}(\alpha_n))$. Then there exists $\widehat{u} \in \big(W^{1,2}(\widehat{\Omega})\big)^2$ and a subsequence of $\{\tilde{u}_n\}$ (denoted by the same symbol) such that*

$$\tilde{u}_n \rightharpoonup \widehat{u} \text{ in } \big(W^{1,2}(\widehat{\Omega})\big)^2$$
$$\tilde{M}_{\alpha_n} D(\tilde{u}_n) \rightharpoonup \tilde{M}_\alpha D(\widehat{u}) \text{ in } \big(L^3(\widehat{\Omega})\big)^{2\times 2}, \; n \to \infty. \qquad (12)$$

In addition, the function $u(\alpha) := \widehat{u}|_{\Omega(\alpha)}$ solves $(\mathcal{P}(\alpha))$ provided that $u(\alpha) \in W_{u_0}(\alpha)$.

Remark 3.2. *Under the assumptions which guarantee uniqueness of the solution to $(\mathcal{P}(\alpha))$ the whole sequence $\{\tilde{u}_n\}$ tends to $\tilde{u}(\alpha)$ in the sense of Theorem 3.1.*

Remark 3.3. *If $W_{u_0}(\alpha) = \widehat{W}_{u_0}(\alpha)$, the assumption $u(\alpha) \in W_{u_0}(\alpha)$ is automatically satisfied.*

Theorem 3.2 (Existence of an optimal shape). *Let there exist a minimizing sequence $\{(\alpha_n, u_n)\}$, $(\alpha_n, u_n) \in \mathcal{G}$, of $(\mathbb{P})$ with an accumulation point (α^*, u^*) such that $u^*|_{\Omega(\alpha^*)} \in W_{u_0}(\alpha^*)$. Then $(\alpha^*, u^*|_{\Omega(\alpha^*)})$ is an optimal pair for $(\mathbb{P})$.*

4 Conclusion

The paper consists of two parts. The first one deals with the existence proof for the generalised steady-state Navier–Stokes system. In the second part the shape optimization problem with the Navier–Stokes system as a state constraint is studied.

Due to an algebraic turbulence model the weak formulation of the state problem involves the weighted Sobolev spaces. The existence and uniqueness of a solution is proved for small data and with a constraint imposed on the model parameters by using energy estimates, the monotone operator theory and the Galerkin method. The analysis of the state problem share many similarities with the techniques presented in Ladyzhenskaya [5, 6], Lions [7] and Parés [10].

The proof of the continuous dependence of solutions on boundary variations is the key result in the shape optimization part. This property is proved under an additional assumption, namely that the limit function of a minimizing sequence belongs to an appropriate space meaning that the existence of an optimal shape is conditional. The paper however suggests a way how to get an unconditional type result.

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