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Time regularity of flows of non-Newtonian fluids

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Abstract: We investigate the higher regularity of the time derivative of a solution to the nonlinear systems describing planar motions of a generalised Newtonian fluid. The system is equipped with homogeneous Dirichlet boundary condition. This condition, together with the nonlinear elliptic term and pressure cause the main difficulty and make the task interesting. The key issue is to show that the second time derivative belongs to $L^\infty_{\mathrm{loc}}(I, L^2(\Omega)) \cap L^2_{\mathrm{loc}}(I, W^{1,2}(\Omega))$. Once we obtain this, the Hölder continuity of time derivative of solution follows. The method is based on a bootstrapping argument in Nikolskii spaces.

Key Words: generalised Newtonian fluids, regularity, evolutionary equations

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, $I := (0,T)$ for some $T > 0$, $Q := I \times \Omega$. We investigate the existence of a regular solution $u : Q \to \mathbb{R}^2$, $\pi : Q \to \mathbb{R}$ of the following two dimensional initial value problem

$$\begin{gathered} \partial_t u + u_i \frac{\partial u}{\partial x_i} - \operatorname{div}(\mathcal{T}(Du)) + \nabla \pi = f, \\ \operatorname{div} u = 0 \text{ in } Q, \\ \int_\Omega \pi(t) = 0 \text{ for a.e. } t \in I, \\ u = u_0 \text{ in } \{0\} \times \Omega \end{gathered} \tag{1}$$

under the homogeneous Dirichlet boundary condition

$$u = 0 \quad \text{on } I \times \partial\Omega. \tag{2}$$

We assume that $\mathcal{T} : \mathbb{S} \to \mathbb{S}$ is of class $C^2_{\mathrm{loc}}(\mathbb{S})$, $\mathbb{S}$ being the set of all symmetric 2×2 matrices, and there exists $p \geq 1$ and $C_1 > 0$ such that for all $D, E \in \mathbb{S}$

$$C_1(1 + |D|^2)^{\frac{p-2}{2}} |E|^2 \leq \partial_{ij} \mathcal{T}_{kl}(D) E_{ij} E_{kl}. \tag{3}$$

In [3, 5] it was proved that under suitable conditions on f, u_0, $\mathcal{T}$ and its growth there exists a weak solution of (1) equipped with periodic or Dirichlet boundary condition which has Hölder continuous space gradient. It is a natural question whether this information allows us to get some better properties of time derivative of u. From Hölder continuity of gradient and (3) it follows that the system can be regarded as a parabolic system with bounded and measurable coefficients. Consequently, the regularity of time derivative of u can be easily obtained provided we consider periodic boundary conditions. Indeed, we can differentiate the equation with respect to time. Multiplying it thereafter with $\partial_t \Delta u$ and integrating over Ω we obtain that $\partial_t u \in L^\infty(I, W^{1,2}(\Omega)) \cap L^2(I, W^{2,2}(\Omega))$. For this step it is decisive that we consider the periodic boundary conditions. In fact, it allows us to test the differentiated equation with $\partial_t \Delta u$ as it satisfies the boundary conditions and consequently the pressure term vanishes as $\partial_t \Delta u$ is solenoidal (compare [5]). Knowing this it is easy to get the information about $\partial_t^2 u$ and all other results from Theorem 1 below in the case of the periodic boundary conditions.

This method unfortunately does not work in the case of Dirichlet boundary conditions because naturally $\partial_t \Delta u$ need not have a zero trace. This situation is normally solved by using a cut off functions as you can see in [7]. But as soon as you use a cut of function the solenoidality of $\partial_t \Delta u$ is ruined and the pressure remains present in the estimates, i.e. we have to know an information about time derivative of pressure (precisely, we need to know $\partial_t \pi$ in $L^2_{\text{loc}}(I, L^2(\Omega))$). Now we get to the heart of the matter. In the system (1) differentiated with respect to time there are namely two problematic terms-the time derivative $\partial_t^2 u$ and the pressure $\nabla \partial_t \pi$. If we know that one of them belongs to $L^2_{\text{loc}}(I, W^{-1,2}(\Omega))$ we get also the same information about the second. By Nečas's theorem on negative norms then follows that $\partial_t \pi$ is locally square integrable in Q. But how to get the initial information? In the case of the Navier-Stokes equations ($T(D) = D$ for all $D \in \mathbb{S}$) the information about $\partial_t^2 u$ is obtained first (even more then necessary) by testing the equation (1) differentiated with respect to time with $\partial_t^2 u$. This is possible due to the fact that when differentiating Laplace operator we obtain again the Laplace operator for derivatives and $\int_\Omega \Delta \partial_t u \partial_t^2 u = \partial_t \|\nabla \partial_t u\|_2^2$. Clearly, this method is not applicable whensoever the tensor $\mathcal{T}$ is nonlinear, we would namely obtain an additional term $\int \partial_D \mathcal{T} |D\partial_t u||D\partial_t^2 u|$ and we did not succeed in estimating it. Similar problems we get also if we try to differentiate (1) twice with respect to time and then test with $\partial_t^2 u$. However, when considering a time differences of (1) instead of time derivatives it appears that it is possible to improve the differentiability of $\partial_t u$ in time. It is shown in Lemma 4. Being aware of this fact, we can then improve regularity of time differences of the solution u on all time levels, see Lemma 5. Iterating this two steps we get even $\partial_t^2 u \in L^\infty_{\text{loc}}(I, L^2(\Omega))$ as it is written in Theorem 1. The whole method is based on the fact that when taking time differences of the system (1) we do not need any cut-off functions in space and when improving the regularity in space we are only on one time layer, also we consider the stationary problem, and there the localisation does not make such troubles as when localising parabolic problem.

2 The main results

Let us assume, that the weak solution u of the problem (1), (2) exists and for a given $q > 2$ satisfies

$$\begin{aligned} u &\in L^\infty_{\text{loc}}(I, W^{2,q}(\Omega)), \\ \partial_t u &\in L^q_{\text{loc}}(I, W^{1,q}(\Omega)) \end{aligned} \tag{4}$$

which is exactly the regularity obtained in [3, 5].

Our main result is the following

Theorem 1 *Let* (3) *and* (4) *hold. Let* $\Omega \in C^2$ *and* $f \in W^{2,\infty}(I, L^2(\Omega))$. *Then*

$$\partial_t^2 u \in L^\infty_{loc}(I, L^2(\Omega)) \cap L^2_{loc}(I, W^{1,2}(\Omega)). \tag{5}$$

Consequently,

$$\nabla \partial_t \pi, \nabla^2 \partial_t u \in L^\infty_{loc}(I, L^2(\Omega)). \tag{6}$$

Moreover, if $f \in W^{2,\infty}(I, L^q(\Omega))$ *then there is* $s > 2$

$$\partial_t^2 u \in L^\infty_{loc}(I, L^s(\Omega)) \cap L^s_{loc}(I, W^{1,s}(\Omega)), \tag{7}$$

$$\partial_t u \in L^\infty_{loc}(I, W^{2,s}(\Omega)) \tag{8}$$

and $\alpha > 0$ *such that*

$$\partial_t u \in C^{0,\alpha}_{loc}(I, C^{1,\alpha}(\overline{\Omega})).$$

Combining the result of Theorem 1 and [3] we get

Theorem 2 *Let* $\mathcal{T}(D) = \nabla_D F(|D|^2)$ *for some* $F : [0, +\infty) \to [0, +\infty)$, $F \in C^3([0, +\infty))$. *Let* (3) *holds with* $p \in [2, 4)$ *and* $|\partial_D \mathcal{T}(D)| \leq C_2(1 + |D|)^{p-2}$ *for some* $C_2 > 0$ *and all* $D \in \mathbb{S}$. *Let* $\Omega \in C^2$, $f \in W^{2,\infty}(I, L^q(\Omega))$ *with* $q > 2$ *and* $u_0 \in L^2(\Omega)$. *Then there is* $\alpha > 0$ *such that the unique weak solution of* (1)-(2) *satisfies* $u \in C^{1,\alpha}_{loc}(I, C^{1,\alpha}(\Omega))$.

In the proof of Theorem 1 we will use Nikolskii spaces $\mathcal{N}^{\alpha,q}(I, X)$, $\alpha \in (0, 1)$, $q \in [1, \infty]$ and X being a Banach space, defined by (compare [8])

$$\mathcal{N}^{\alpha,q}(I, X) = \{g \in L^q(I, X) : \\ \sup_{h>0} h^{-\alpha} \|f(\cdot) - f(\cdot - h)\|_{L^q(I_h, X)} < +\infty\}.$$

We defined $I_h := \{t \in I : t - h \in I\}$. For simplicity we identify $\mathcal{N}^{0,q}(I, X) = L^q(I, X)$ and $\mathcal{N}^{1,q}(I, X) = W^{1,q}(I, X)$.

3 Proof of Theorem 1

Before we prove the theorem, we state some auxiliary lemmas. Setting $h \in (0, T)$ we may define $w(t, x) := u(t, x) - u(t - h, x)$ and $\sigma(t, x) := \pi(t, x) - \pi(t - h, x)$ if $t > h$. For the so-defined differences the following lemma holds.

Lemma 3 *Let*

$$\begin{aligned} F(t, x) :=& f(t, x) - f(t - h, x) \\ &+ (u \cdot \nabla)u(t, x) - (u \cdot \nabla)u(t - h, x), \\ A(t, x) :=& \int_0^1 \partial_D \mathcal{T}(sDu(t, x) \\ &+ (1 - s)Du(t - h, x))ds. \end{aligned}$$

The equation

$$\partial_t w - \operatorname{div}(ADw) + \nabla\sigma = F, \quad \operatorname{div} w = 0 \qquad (9)$$

is satisfied pointwise almost everywhere in $(h,T)\times\Omega$ *and* $w(t) = 0$ *on* $\partial\Omega$ *in the sense of traces for a.e.* $t \in (h,T)$.

The validity of the lemma follows easily from (3) and (4). At this moment it is worth noting that for $A \in L^\infty(Q)$ there exists by (3) and (4) $\gamma_2 \geq \gamma_1 > 0$ such that for all $E \in \mathbb{S}$

$$\gamma_1 |E|^2 \leq AE \otimes E \leq \gamma_2 |E|^2 . \qquad (10)$$

Moreover, by (4) and the fact that $\mathcal{T} \in C^2_{\text{loc}}(\mathbb{S})$

$$|\partial_t A| \leq C(|D\partial_t u(t)| + |D\partial_t u(t-h)|). \qquad (11)$$

The first of the main ideas of the proof of Theorem 1 is that from (9) we can slightly improve regularity of u in time.

Lemma 4 *Let* $\alpha \in [0,1]$ *and* $r > 1$ *such that* $1 = 1/2 + 1/q + 1/r$. *From*

$$u \in \mathcal{N}^{\alpha,\infty}_{loc}(I, W^{2,2}(\Omega)) \qquad (12)$$

it follows that

$$\partial_t u \in \mathcal{N}^{\frac{2}{r}+(1-\frac{2}{r})\alpha,\infty}_{loc}(I, L^2(\Omega)), \qquad (13)$$

$$\partial_t u \in \mathcal{N}^{\frac{2}{r}+(1-\frac{2}{r})\alpha,2}_{loc}(I, W^{1,2}(\Omega)), \qquad (14)$$

$$u \in \mathcal{N}^{\frac{2}{r}+(1-\frac{2}{r})\alpha,\infty}_{loc}(I, W^{1,2}(\Omega)). \qquad (15)$$

Proof. To remove any possible problems with localising in time we follow [6]. First we multiply (9) by $\partial_t w$ and integrate it over Ω to obtain for all times $t \in (h,T)$

$$\|\partial_t w\|_2^2 \leq C\int_\Omega (|F\partial_t w| + |ADwD\partial_t w|). \qquad (16)$$

The pressure term vanished due to $\operatorname{div}\partial_t w = 0$.

In the next step we differentiate (9) with respect to time. The so-obtained equation we multiply by $\partial_t w$ and integrate over Ω. Using $\operatorname{div}\partial_t w = 0$ to eliminate the pressure term and properties (10), (11) of A we get

$$\partial_t \|\partial_t w\|_2^2 + \|D\partial_t w\|_2^2 \leq C\int_\Omega \partial_t F \partial_t w + (|D\partial_t u(t)| + |D\partial_t u(t-h)|)|Dw||D\partial_t w|. \qquad (17)$$

As the equations (16) and (17) make sense only for $t > h$ we multiply (16) by $\partial_t\eta$ and (17) by η (η is a suitable cut off function in time) and sum them together. On the right hand side we obtain many terms but we mention only the worst of them

$$\partial_t(\eta\|\partial_t w\|_2^2) + \eta\|D\partial_t w\|_2^2 \leq C\eta\int_\Omega (|D\partial_t u(t)| + |D\partial_t u(t-h)|)|Dw||D\partial_t w| + C\eta\int_\Omega |\partial_t u(t-h)||\nabla w|\ |\partial_t w|. \qquad (18)$$

The origin of the first term on the right is clear and the second one is a representative of terms which are hidden in (17) in $\partial_t F\partial_t w$. Let us now estimate the first term on the right hand side by Young's inequality

$$\eta\int_\Omega |D\partial_t u||Dw||D\partial_t w| \leq \eta(\frac{1}{2}\|D\partial_t w\|_2^2 + C\|D\partial_t u\|_q^q + C\|Dw\|_r^r).$$

By interpolation $\|Dw\|_r^r \leq C\|Dw\|_2^2 \|Dw\|_{1,2}^{r-2}$ we get, estimating

$$\int_I \eta\|Dw\|_2^2 \leq Ch^2$$

by (4) and

$$\sup_{\operatorname{supp}\eta} \|Dw\|_{1,2}^{r-2} \leq Ch^{\alpha(r-2)}$$

by (12), that

$$\int_I \eta\|Dw\|_r^r \leq C\int_I \eta\|Dw\|_2^2\|Dw\|_{1,2}^{r-2} \leq Ch^{r(\frac{2}{r}+\frac{r-2}{r}\alpha)}.$$

Concerning the second term on the right hand side of (18) we consider (4) and estimate it with help of embedding theorem, Hölder's and Young's inequality as follows

$$\eta\int_\Omega |\partial_t u(t-h)||\nabla w|\ |\partial_t w| \leq \eta\|\partial_t u\|_\infty \|\nabla w\|_2 \|\partial_t w\|_2 \leq C\|\partial_t u\|_{1,q}^2 + \eta\|\nabla w\|_2^2\|\partial_t w\|_2^2 .$$

We see that if we divide equation (18) by $h^{2(\frac{2}{r}+(1-\frac{2}{r})\alpha)}$ and denote $\tilde{w} = w/h^{\frac{2}{r}+(1-\frac{2}{r})\alpha}$ we get after repeating the above calculations and integration from 0 to $t \in (h,T)$ that

$$\|\partial_t \tilde{w}\|_2^2(t) + \int_0^t \eta \|D\partial_t \tilde{w}\|_2^2 \leq$$
$$C(1 + \int_0^T \eta \|\partial_t u\|_{1,q}^q + \int_0^t \eta \|\nabla \tilde{w}\|_2^2 \|\partial_t \tilde{w}\|_2^2).$$

Now it is enough to realize that due to (4) $\int_I \eta \|\nabla w\|_2^2 \leq Ch^2$ and that $\partial_t u \in L^q_{\mathrm{loc}}(I, W^{1,q}(\Omega))$. Gronwall's inequality than implies the statements (13) and (14). Since

$$\partial_t(\eta \|Dw\|_2^2) \leq C(\|Dw\|_2^2 + \|\partial_t Dw\|_2^2)$$

by Young's inequality, (15) is a consequence of (4) and (14). The lemma is proved. □

The second milestone of the proof of Theorem 1 is that after the improvement of the regularity in time, we can improve also regularity in space on single time layers.

Lemma 5 *Let $\beta \in [0,1]$. If*

$$\begin{aligned} \partial_t u &\in \mathcal{N}^{\beta,\infty}_{loc}(I, L^2(\Omega)), \\ u &\in \mathcal{N}^{\beta,\infty}_{loc}(I, W^{1,2}(\Omega)) \end{aligned} \tag{19}$$

then

$$u \in \mathcal{N}^{\beta,\infty}_{loc}(I, W^{2,2}(\Omega)). \tag{20}$$

Proof. To prove this lemma we note, that under its assumptions for all $t \in J$ ($J \subset \overline{J} \subset I$)

$$\|G(t)\|_2 := \|F(t) - \partial_t w(t)\|_2 \leq Ch^\beta, \tag{21}$$

and use the difference technique in space for all time levels $t \in J$. As we deal with the Dirichlet boundary condition it is technically difficult, especially in the neighbourhood of $\partial\Omega$. Nevertheless, it was described in detail for example in [4] or [7] and it works in our case in the same manner. That's why we only sketch the proof here. The main idea is to differentiate the equation (9) in some space direction, we denote this differentiation with an apostrophe, and test with w'. Note that we have enough regularity to write all integrals because of (4), but though we are not allowed to test with w' since it does not satisfy the boundary conditions. Consult the technique how to avoid this problems with [4]. We just want to emphasise that it is important that in the moment we handle the stationary equation. The method from [4] namely doesn't work well in the case of parabolic equations. Also just informally

$$\begin{aligned} \int_\Omega ADw'Dw' \leq C \int_\Omega (|Gw''| \\ + (|Du'(t+h)| + |Du'(t)|)|Dw||Dw'|). \end{aligned} \tag{22}$$

The first term on the right we estimate ($\epsilon > 0$ small)

$$\int_\Omega |Gw''| \leq \|G\|_2^2 + \epsilon \|Dw'\|_2^2$$

and the second one ($\epsilon > 0$ small)

$$\int_\Omega |Du'||Dw||Dw'| \leq$$
$$C\|Du'\|_q^2 \|Dw\|_r^2 + \epsilon \|Dw'\|_2^2.$$

Since $\|Du'\|_q$ is bounded in J by (4) and $\|Dw\|_r^2 \leq \epsilon \|D^2 w\|_2^2 + C\|Dw\|_2^2$ by Ehrling's lemma we get by (19) and (21) that

$$(\text{RHS of (22)}) \leq \epsilon \|D^2 w\|_2^2 + Ch^{2\beta}.$$

Then (20) follows from (22) by (10). □

Finally we are ready to prove the main theorem.

Proof of Theorem 1. To prove the theorem we use a bootstrap argument. Let us define $\alpha_1 = 0$, $\alpha_{i+1} = 2/r + (1-2/r)\alpha_i$ for all $i \in \mathbb{N}$. Then $\{\alpha_i\}_{i=1}^\infty$ is a monotone sequence converging to 1. Since $u \in \mathcal{N}^{0,\infty}_{\mathrm{loc}}(I, W^{2,2}(\Omega))$ by (4) and from Lemmas 4 and 5 it follows for all $i \in \mathbb{N}$ that if $u \in \mathcal{N}^{\alpha_i,\infty}_{\mathrm{loc}}(I, W^{2,2}(\Omega))$ then $u \in \mathcal{N}^{\alpha_{i+1},\infty}_{\mathrm{loc}}(I, W^{2,2}(\Omega))$, we have that for all $i \in \mathbb{N}$

$$\begin{aligned} u &\in \mathcal{N}^{\alpha_i,\infty}_{\mathrm{loc}}(I, W^{2,2}(\Omega)), \\ \partial_t u &\in \mathcal{N}^{\alpha_i,2}_{\mathrm{loc}}(I, W^{1,2}(\Omega)). \end{aligned}$$

Let $J \subset \overline{K} \subset I$, $h > 0$ such that for all $t \in J$ it is $t+h \in K$ and $t-h \in K$. Denoting now $\tilde{w}(t) = u(t+h) - 2u(t) + u(t-h)$ the second time difference of u, it follows from $u \in \mathcal{N}^{\alpha_i,\infty}_{\mathrm{loc}}(I, W^{2,2}(\Omega))$ that

$$\text{ess-sup}_J \|\tilde{w}\|_{2,2} \leq Ch^{\alpha_i}. \tag{23}$$

Similarly, we have from $\partial_t u \in \mathcal{N}^{\alpha_i,2}_{\mathrm{loc}}(I, W^{1,2}(\Omega))$ that

$$(\int_J \|\tilde{w}\|_{1,2}^2)^{1/2} \leq Ch^{\alpha_i+1}. \tag{24}$$

Standard interpolation together with (23) and (24) gives

$$\int_J \|\tilde{w}\|_{1,4}^4 \le \int_J \|\tilde{w}\|_{1,2}^2 \|\tilde{w}\|_{2,2}^2 \le Ch^{4\alpha_i+2}.$$

Consequently, choosing $i \in \mathbb{N}$ such that $\alpha_i > 1/2$, by Reduction theorem, see [1, Section 5.4] $\partial_t u \in \mathcal{N}_{\text{loc}}^{\alpha_i-1/2,4}(I, W^{1,4}(\Omega))$ and clearly also $\partial_t u \in L^4_{\text{loc}}(I, W^{1,4}(\Omega))$. Knowing this it is easy to get

$$\text{ess-sup}_J \|\partial_t w\|_2^2 (t) + \int_J \|D\partial_t w\|_2^2 \le Ch^2$$

from (18) and (5) follows.

With help of (5) we can realize that the assumptions of Lemma 5 are fulfilled with $\beta = 1$. It follows that $\nabla^2 \partial_t u \in L^\infty_{\text{loc}}(I, L^2(\Omega))$. Differentiating (1) with respect to time

$$\begin{aligned} \partial_t^2 u - &\operatorname{div}(\partial_D \mathcal{T}(Du) D\partial_t u) + \nabla \partial_t \pi \\ &= \partial_t (f - u \cdot \nabla u) \end{aligned} \tag{25}$$

we reread from (25) that $\nabla \partial_t \pi \in L^\infty_{\text{loc}}(I, L^2(\Omega))$, hence (6) is proved.

To prove (7) and (8) one can use the method from the article [3, Section 3] for the equation (25). The scheme of the proof is the following. First (7) is shown. Consequently, $\partial_t^2 u$ is moved to the right hand side of (25) and the elliptic theory is applied at almost every time level. Both these steps are based on using the L^q-theory for the generalised Stokes problem (see [3, 4]) to (25) differentiated with respect to time (in the first case) or in space (in the second case). Note that the situation is slightly complicated than in [3, Section 3], since we obtain some additional terms when differentiating the elliptic term. We sketch only how to estimate these terms. When getting the information about $\partial_t^2 u$ we differentiate (25) with respect to time. The additional term we get from elliptic term is $\partial_D^2 \mathcal{T}(Du) D\partial_t u D\partial_t u$ which belongs to $L^\infty_{\text{loc}}(I, L^s(\Omega))$ for all $s > 1$ due to (6). So we can proceed as in [3, Lemma 3.3] to get (7). Knowing this we move $\partial_t^2 u$ in (25) to the right and on a fixed time level use regularity theory for elliptic problems, see [4]. Also here an unpleasant term arising when differentiating the elliptic one appears. It is $\partial_D^2 (Du) D^2 u D\partial_t u$ which however belongs to $L^\infty_{\text{loc}}(I, L^s(\Omega))$ for some $s > 2$ thanks to (4) and (6), and (8) follows in the same way as in [4]. The Hölder continuity of $\partial_t u$ is a consequence of (7), (8) and the embedding [2, Lemma 2.2]. The theorem is proved. □

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