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Analytical Solution to Slow Flow of a Cosserat Fluid Past a Sphere Using Slip Boundary Condition

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Abstract: - In the present paper the theory of the micropolar fluids based on the theory of Cosserat continua has been applied for analysis of Stokes flow (i.e., creeping flow) past a rigid sphere. The flow is considered in the presence of Trostel's slip boundary condition which states that the tangential velocity on the solid wall (i.e. slip velocity) is proportional to the shear stress on the wall. The solution of problem in this case is reduced to simultaneous solution of biharmonic and Helmholtz equations for Stokes stream function. The problem is solved by the method of separation of variables in spherical coordinates. The solution is used to plot streamline topology of this flow and the pressure and drag coefficients are studied as the slip parameter is varying.

Key-Words: - Stokes Flow-COSSERAT Continuum-Micropolar Fluid-Sphere-Slip Boundary Condition

1 Introduction

The concept of Cosserat continua was introduced in a paper submitted by two French brothers E. and F. Cosserat in 1909 [1]. In this continuum, we consider the effect of couples on a material element in addition to and independent of the effect of forces.

The theory of micro-fluids, introduced by A. C. Eringen [2,3,4], deals with a class of fluids which exhibit certain microscopic effects arising from the local structure and micromotions of fluid element. A subclass of these is the micropolar fluid which has the microrotational effects and microrotational inertia [3].

This class of fluids can support the couple stress, the body couples and the non-symmetric stress tensor and possess a rotation field, which is independent of the velocity field. The rotation field is no longer equal to the half of the curl of the velocity vector field. Because of the assumption of infinitesimal rotations, we can treat the rotation field as a vector field.

The theory, thus, has two independent kinematic variables; the velocity vector $\mathbf{V}$ and the spin or microrotation vector $\boldsymbol{\nu}$.

The linear constitutive equation for non-symmetric stress tensor (i.e., Cauchy's stress tensor), contains an additional viscosity coefficient k_v. The value of k_v shows the influence of the microrotation field on the stress tensor.

The linear constitutive equation for couple stress also contains three additional viscosity coefficients α_v, β_v and γ_v.

The slippage of fluids on solid walls was a challenging discussion among fluid dynamists during the history of development of this science. There are 3 ideas about boundary condition of fluid on solid walls in 19th century:

The fluid adheres the wall and the velocity is growing continuously from boundary (i.e. the so-called no-slip condition)

A layer of fluid with finite width remains stationary on the wall and the remainder of fluid slips on this layer.

The fluid is slipping on wall, which is known as slip boundary condition.

Finally, Stokes stated that the "the slip doesn't exist" and ended the discussions. But Trostel starts the discussion about this matter again in 1988. He proved that fluids slip on solid walls [5,6,7]. He proposed that the tangential velocity on the wall is proportional to the value of the shear stress, namely

$$\tau\big|_{\text{Boundary}} = \lambda\, V_t\big|_{\text{Boundary}}\,, \tag{1}$$

where λ is the positive scalar coefficient of surface with the unit of $\text{Pa.sec}/\text{m}$ and V_t denotes the tangential velocity (i.e., slip velocity).

It is a complete boundary condition which covers the full slip boundary condition for $\lambda = 0$ and the no-slip one for $\lambda \to \infty$.

2 Motion of a COSSERAT Fluid

2.1 Kinematics of COSSERAT Continua

At any material point of the continuum, we consider both a velocity and a rotation velocity vector denoted by $\mathbf{V}$ and $\boldsymbol{\nu}$, respectively. The so-called COSSERAT microrotation R_{ij} relates the current state of a triad of orthonormal directions attached to each material point to its initial state, i.e.

$$R_{ij} = \delta_{ij} - \Gamma_{ijk}\nu_k, \tag{2}$$

where δ_{ij} and Γ_{ijk} are the Kronecker delta tensor and permutation tensor, respectively.

The associated COSSERAT deformation ε_{ij} and torsion-curvature tensor κ_{ij} are written as

$$\varepsilon_{ij} = V_{j,i} - \Gamma_{ijk}\nu_k, \tag{3}$$

$$\kappa_{ij} = \nu_{j,i}, \tag{4}$$

where the comma denotes the partial differentiation.

2.2 Balance Laws in COSSERAT Media

It is assumed that the transfer of interaction between two particles of the continuum through a surface element $n_i ds$ occurs by means of both a traction vector $t_i ds$ and a moment vector $m_i ds$. Surface forces and couples are represented by the generally non-symmetric force-stress and couple-stress tensors t_{ij} and m_{ij}, respectively.

The axioms of balance of linear momentum and moment of momentum (i.e. angular momentum) require that the following equations hold

$$t_{ij,j} + f_i = \rho\frac{DV_i}{Dt}, \tag{5}$$

$$m_{ji,j} + \Gamma_{ijk}t_{ik} + l_i = j\frac{D\nu_i}{Dt}, \tag{6}$$

where ρ, j, f_i and l_i are the mass density, microinertia, body force per unit mass and body couple per unit mass respectively.

2.3 Constitutive Equations

Here we choose linear constitutive equations which describe our material behavior. It can be considered as the generalization of Newtonian fluids in the classical Navier-Stokes theory.

$$\begin{aligned} t_{kl} &= \left(-\pi + \lambda_v V_{r,r}\right)\delta_{kl} + \mu_v\left(V_{k,l} + V_{l,k}\right) \\ &\quad + k_v\left(V_{l,k} - \Gamma_{klr}\nu_r\right), \end{aligned} \tag{7}$$

$$m_{kl} = \alpha_v \nu_{r,r}\delta_{kl} + \beta_v \nu_{k,l} + \gamma_v \nu_{l,k}, \tag{8}$$

where π is the thermodynamic pressure.

As you see, the microrotation field has influence on the stress tensor, but the vice versa is not true.

2.4 Field Equations

At this stage we must mix the above equations to obtain governing field equations. The field equations for micropolar fluids in the vectorial form are given by

Conservation of mass (i.e. continuity equation)

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\left(\rho\mathbf{V}\right) = 0, \tag{9}$$

Balance of momentum

$$\begin{aligned} &\left(\lambda_v + 2\mu_v + k_v\right)\nabla\nabla\cdot\mathbf{V} - \left(\mu_v + k_v\right)\nabla\times\nabla\times\mathbf{V} \\ &+ k_v\nabla\times\boldsymbol{\nu} - \nabla\pi + \rho\left(\mathbf{f} - \frac{D\mathbf{V}}{Dt}\right) = \mathbf{0}, \end{aligned} \tag{10}$$

Balance of moment of momentum

$$\begin{aligned} &\left(\alpha_v + \beta_v + \gamma_v\right)\nabla\nabla\cdot\boldsymbol{\nu} - \gamma_v\nabla\times\nabla\times\boldsymbol{\nu} \\ &+ k_v\nabla\times\mathbf{V} - 2k_v\boldsymbol{\nu} + \rho\left(\mathbf{l} - j\frac{D\boldsymbol{\nu}}{Dt}\right) = \mathbf{0}, \end{aligned} \tag{11}$$

where $\frac{D}{Dt}$ denotes the material time derivative.

3 Boundary conditions

By their character, the boundary conditions can be twofold: kinematic and dynamic. The kinematic boundary conditions lie in the fact that the kinematic variables, the velocity and the angular velocity have definite quantities at the boundary. In the dynamic boundary conditions, however, on the boundary surfaces, the values of the stress and the couple-stress are fixed.

From the mathematical point of view, the kinematic boundary condition is a Dirichlet's one and the dynamic boundary condition is a von Neumann's one.

In the present work, we have kinematic boundary conditions for velocity of free-stream at infinity and for microrotation on sphere wall. But for velocity on sphere wall, we use the Trostel's slip boundary condition which is a linear combination of kinematic and dynamic boundary conditions. Thus it is a Rubin's boundary condition. It shows that slip boundary condition is kinematic and dynamic, but the slip theory is thermodynamic in nature.

4 The Stokes flow about a sphere

In classical approach to fluid mechanics, the governing equations of fluid flow (i.e., the Navier-Stokes equations) are nonlinear. Their nonlinearity is geometrical and is due to the kinematics of flow.

So they don't have analytical solution except for very simple problems. To avoid this difficulty, it is conventional in fundamental mechanics of fluids to linearize these equations. Since the nonlinearity is due to inertial terms, Stokes assumed that in very slow motion of fluids about immersed bodies, the inertial effects can be neglected. By this simplification, a linear system of equations is obtained and it has some analytical closed-form solutions.

In this paper, we have done such a simplification for equations of motion obtained from Cosserat theory and we also call it as Stokes flow.

The solutions obtained by the above assumption, are limited to the case of very low Reynolds number flows.

Here we consider the slow steady flow of an incompressible micropolar fluid around a sphere of radius a, held fixed. The uniform stream speed at infinity is taken to be V_∞. External loads are considered absent. In spherical coordinates (r,θ,ϕ), taking $\theta=0$ as the axis in the direction of the free-stream flow, the velocity $\mathbf{V}$ and microrotation $\boldsymbol{\nu}$ possess axial symmetry so that

$$\mathbf{V}=(V_r,V_\theta,0) \quad , \quad \boldsymbol{\nu}=(0,0,\nu). \tag{12}$$

Stokes stream function is defined by

$$V_r=-\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} \quad , \quad V_\theta=\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}. \tag{13}$$

Substituting these into the field equations (10) and (11) leads to

$$\Delta\Delta\left(\Delta-\xi^2\right)\psi=0, \tag{14}$$

$$\nu=\frac{1}{2r\sin\theta}\left(\Delta\psi+\frac{\gamma_v(\mu_v+k_v)}{k_v^2}\Delta\Delta\psi\right), \tag{15}$$

where the Laplacian operator $\Delta=\nabla^2$ is defined by

$$\Delta=\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}-\frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}. \tag{16}$$

Equations (14) and (15) are subjected to the boundary conditions

$$\frac{\partial\psi}{\partial\theta}=0, \quad t_{r\theta}=\lambda V_\theta \quad \text{at } r=a,$$

$$\nu=0 \quad \text{at } r=a, \tag{17}$$

$$\psi\to\frac{1}{2}V_\infty r^2\sin^2\theta \quad \text{as } r\to\infty.$$

5 Solution to Problem

The solution is obtained by taking

$$\psi=\psi_1+\psi_2, \tag{18}$$

where ψ_1 and ψ_2 satisfy the equations

$$\Delta\Delta\psi_1=0, \tag{19}$$

$$\left(\Delta-\xi^2\right)\psi_2=0 \quad , \quad \xi^2\equiv\frac{k_v(2\mu_v+k_v)}{\gamma_v(\mu_v+k_v)}. \tag{20}$$

The solution of (19) is well known in the classical case (Newtonian fluid). The solution of (20) is obtained similarly so that

$$\psi_1=\frac{V_\infty}{2}\sin^2\theta\left(\frac{A_1}{r}+B_1r+C_1r^2+D_1r^4\right), \tag{21}$$

$$\psi_2=\frac{V_\infty}{2}\sin^2\theta\left[A_2\left(\xi-\frac{1}{r}\right)e^{\xi r}+B_2\left(\xi+\frac{1}{r}\right)e^{-\xi r}\right]. \tag{22}$$

Both the solutions are obtained by using the method of separation of variables.

The condition at infinity demands that $D_1=A_2=0$ and $C_1=1$. Hence

$$\psi=\frac{V_\infty}{2}\sin^2\theta\left[\frac{A_1}{r}+B_1r+r^2+B_2\left(\xi+\frac{1}{r}\right)e^{-\xi r}\right]. \tag{23}$$

We can write (15) as

$$\nu=\frac{V_\infty\sin\theta}{2r}\left[\frac{B_2(\mu_v+k_v)}{k_v}\xi^2\left(\xi+\frac{1}{r}\right)e^{-\xi r}-\frac{B_1}{r}\right]. \tag{24}$$

The velocity components are given by (13)

$$V_r=\frac{V_\infty\cos\theta}{r^2}\left[\frac{A_1}{r}+B_1r+r^2+B_2\left(\xi+\frac{1}{r}\right)e^{-\xi r}\right], \tag{25}$$

$$V_\theta=\frac{V_\infty\sin\theta}{2r}\left[\begin{array}{l}-\frac{A_1}{r^2}+B_1+2r\\ -B_2\left[\xi\left(\xi+\frac{1}{r}\right)+\frac{1}{r^2}\right]e^{-\xi r}\end{array}\right]. \tag{26}$$

The pressure field and the normal and shear stress components are obtained from constitutive equation (7) in spherical coordinates.

$$p=p_\infty-\frac{2\mu_v+k_v}{2r^2}B_1V_\infty\cos\theta, \tag{27}$$

$$t_{rr}=-p_\infty+\frac{2\mu_v+k_v}{r^2}V_\infty\cos\theta \left[\frac{3}{2}B_1+\frac{3A_3}{r^2}+B_2\left(\xi^2+\frac{3\xi}{r}+\frac{3}{r^2}\right)e^{-\xi r}\right], \tag{28}$$

$$t_{r\theta} = \frac{2\mu_v + k_v}{2r^2} V_\infty \sin\theta \left[\frac{3A_1}{r^2} + B_2 \left(\xi^2 + \frac{2\xi}{r} + \frac{2}{r^2} \right) e^{-\xi r} \right], \tag{29}$$

Now, by applying the boundary conditions (17) on the sphere wall, for stream function and microrotation, the remainder integration constants are determined.

$$A_1 = \frac{a}{\xi^2 \left\{ \begin{array}{l} \zeta^2 (4+3\chi) + 2(1+\chi)(3+\mathrm{Tr}) \\ +\zeta\left[\mathrm{Tr}(3+2\chi) + (11+9\chi)\right] \end{array} \right\}} \times \left\{ \begin{array}{l} -(3+3\chi+\chi^2)\zeta^2 + \mathrm{Tr}\chi^2(1+\chi) \\ +\zeta\left[\mathrm{Tr}(-3-3\chi+\chi^3) - 2(3+3\chi+\chi^2)\right] \end{array} \right\}, \tag{30}$$

$$B_1 = -\frac{3a(1+\chi)(1+\zeta)(\mathrm{Tr}+\zeta+2)}{\left\{ \begin{array}{l} (4+3\chi)\zeta^2 + 2(1+\chi)(3+\mathrm{Tr}) \\ +\zeta\left[\mathrm{Tr}(3+2\chi) + (11+9\chi)\right] \end{array} \right\}}, \tag{31}$$

$$B_2 = \frac{3ae^{\chi}\zeta(\mathrm{Tr}+\zeta+2)}{\xi^2 \left\{ \begin{array}{l} (4+3\chi)\zeta^2 + 2(1+\chi)(3+\mathrm{Tr}) \\ +\zeta\left[\mathrm{Tr}(3+2\chi) + (11+9\chi)\right] \end{array} \right\}}, \tag{32}$$

where $\zeta = k_v/\mu_v$ and $\chi = a\xi$.

The drag on the sphere is obtained by using the integral formula

$$D = 2\pi a^2 \int_0^\pi \left(t_{rr}\cos\theta - t_{r\theta}\sin\theta \right)_{r=a} \sin\theta \, d\theta. \tag{33}$$

As you see, the drag is formed from two parts; a part due to normal stress, D_{rr} and another due to shear stress, $D_{r\theta}$.

$$D = \frac{6\pi a \mu_v V_\infty (1+\chi)(1+\zeta)(2+\zeta)(\mathrm{Tr}+\zeta+2)}{\left\{ \begin{array}{l} (4+3\chi)\zeta^2 + 2(1+\chi)(\mathrm{Tr}+3) \\ +\zeta\left[\mathrm{Tr}(3+2\chi) + (11+9\chi)\right] \end{array} \right\}}, \tag{34}$$

where $\mathrm{Re} = \dfrac{2\rho a V_\infty}{\mu_v}$.

The drag coefficient for very slow motion of a Cosserat fluid about a sphere with Trostel's slip boundary condition is given by

$$C_D = \frac{D}{\frac{1}{2}\rho V_\infty^2 A} = \frac{24}{\mathrm{Re}} \left[\frac{(1+\chi)(1+\zeta)(2+\zeta)(\mathrm{Tr}+\zeta+2)}{\left\{ \begin{array}{l} (4+3\chi)\zeta^2 + 2(1+\chi)(\mathrm{Tr}+3) \\ +\zeta\left[\mathrm{Tr}(3+2\chi) + (11+9\chi)\right] \end{array} \right\}} \right]. \tag{35}$$

where $A = \pi a^2$

This is reduced to the result of classical fluid mechanics by setting $\chi = \zeta = 0$, which is

$$C_D = \left(\frac{\mathrm{Tr}+2}{\mathrm{Tr}+3} \right) \frac{24}{\mathrm{Re}}. \tag{36}$$

For two limiting case $\mathrm{Tr} = 0$ (i.e., full slip) and $\mathrm{Tr} \to \infty$ (i.e., no-slip) we have this results from classical theory

$$C_{D,\text{classical,no-slip}} = \frac{24}{\mathrm{Re}}, \tag{37}$$

$$C_{D,\text{classical,full slip}} = \frac{16}{\mathrm{Re}} = \frac{2}{3} C_{D,\text{classical,no-slip}}. \tag{38}$$

Now we are able to calculate the amount of effect of normal stress and shear stress on drag of a sphere

$$\frac{D_{rr}}{D} = \frac{\mathrm{Tr}+3\zeta+6}{3\mathrm{Tr}+3\zeta+6} \Rightarrow \begin{cases} \dfrac{D_{rr}}{D} = 1, & \text{for } \mathrm{Tr} = 0, \\ \dfrac{D_{rr}}{D} = \dfrac{1}{3}, & \text{for } \mathrm{Tr} \to \infty, \end{cases} \tag{39}$$

$$\frac{D_{r\theta}}{D} = \frac{2\mathrm{Tr}}{3\mathrm{Tr}+3\zeta+2} \Rightarrow \begin{cases} \dfrac{D_{r\theta}}{D} = 0, & \text{for } \mathrm{Tr} = 0, \\ \dfrac{D_{r\theta}}{D} = \dfrac{2}{3}, & \text{for } \mathrm{Tr} \to \infty. \end{cases} \tag{40}$$

As you see, the induced drag in full slip condition is only due to normal stress, because in this case the value of shear stress on wall is zero.

The pressure coefficient on sphere wall is given by

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho V_\infty^2} = -\frac{2\mu_v + k_v}{a^2} B_1 \frac{\cos\theta}{\rho V_\infty} = -\mathrm{Re}_\mu B_1 \frac{8\cos\theta}{a}, \tag{41}$$

where $Re_\mu = \dfrac{4\rho V_\infty a}{2\mu_v + k_v}$ is Reynolds number with respect to both μ_v and k_v.

6 Conclusion

In the present paper the theory of the micropolar fluid based on a Cosserat continuum model has been applied to an actual problem. We change the governing equations either the boundary conditions in comparison to classical fluid mechanics. This work leads to new terms in results such as stream function, velocity components, drag and pressure coefficients. We demonstrated that the slippage of fluid on sphere wall reduces the induced drag slightly. We also showed that the solution reduced to classical solution by equating all constants due to Cosserat theory to zero.

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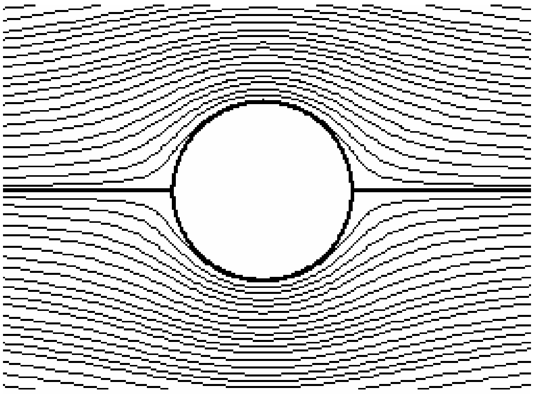

Fig 1, Streamlines
Re=1, Tr=∞

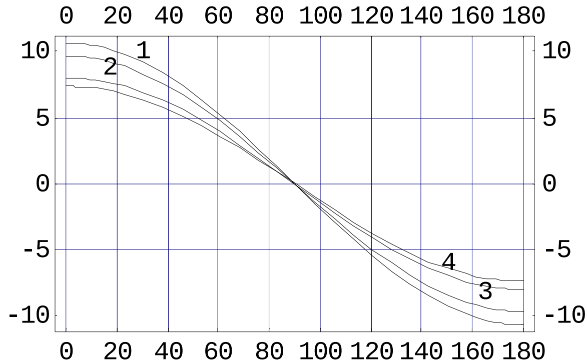

Fig 2. Pressure Coefficient
$Re_{\mu} = 1, \zeta = 1, \chi = 1$
1:Tr=∞, 2:Tr=10, 3:Tr=1, 4:Tr=0