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Numerical solution of inviscid compressible flow with low Mach numbers

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Abstract: The paper is concerned with the numerical solution of inviscid compressible flow described by the system of the Euler equations. Our goal is to work out a numerical scheme robust with respect to the magnitude of the Mach number, i.e. a scheme applicable to flows with a wide range of Mach numbers, from very low Mach number flow up to hypersonic regimes. Our method is based on the application of the discontinuous Galerkin finite element method to space discretization, semi-implicit time discretization and characteristics treatment of boundary conditions. As numerical tests show, the method is sufficiently accurate, efficient and robust and allows to solve compressible flow with practically all Mach numbers without any modification of the Euler equations.

Key Words: discontinuous Galerkin finite element method, numerical flux, semi-implicit time discretization, GMRES method, method of characteristics, low Mach number flow

1 Introduction

In the numerical solution of compressible flow, it is necessary to overcome a number of obstacles. Let us mention the necessity to resolve accurately shock waves, contact discontinuities and (in viscous flow) boundary layers, wakes and their interaction. All these phenomena are connected with the simulation of high speed flow with high Mach numbers. However, it appears that the solution of low Mach number flow is also rather difficult. This is caused by the stiff behaviour of numerical schemes and acoustic phenomena appearing in low Mach number flows at incompressible limit. In this case, standard finite volume schemes fail. This led to the development of special finite volume techniques allowing the simulation of compressible flow at incompressible limit, which is based on modifications of the Euler or Navier-Stokes equations. (See, e.g. [8], [11], [13], Chapter 14, or [10], Chapter 5.)

Our goal is to develop a numerical technique allowing the solution of compressible flow with a wide range of the Mach number without any modification of the governing equations. This technique is based on the *discontinuous Galerkin finite element method* (DGFEM), which can be considered as a generalization of the finite volume as well as finite element methods, using advantages of both these techniques. It employs piecewise polynomial approximations without any requirement on the continuity on interfaces between neighbouring elements. The discontinuous Galerkin space semidiscretization is combined with a semi-implicit time discretization and a special treatment of boundary conditions in inviscid convective terms. In this way we obtain a numerical scheme requiring the solution of only one linear system on each time level.

In Section 2 the continuous problem describing inviscid compressible flow is formulated. In Section 3 the discontinuous Galerkin space semidiscretization is introduced. Further, in Section 4 a semi-implicit time discretization is developed. Section 5 is concerned with the treatment of boundary conditions. Finally, in Section 6 we present an interesting example of the DGFE solution of an inviscid compressible flow past a circular cylinder at incompressible limit.

The computational results show that the presented method is applicable to the numerical solution of inviscid compressible flow with a very low Mach number at incompressible limit.

2 Continuous problem

For simplicity of the treatment we shall consider 2-dimensional flow, but the method can be applied to 3D flow as well. The system of the Euler equations describing 2D inviscid flow can be written in the form

$$\frac{\partial \boldsymbol{w}}{\partial t}+\sum_{s=1}^{2}\frac{\partial \boldsymbol{f}_s(\boldsymbol{w})}{\partial x_s}=0 \quad \text{in } Q_T=\Omega\times(0,T), \quad (1)$$

where $\Omega\subset I\!R^2$ is a bounded domain occupied by gas, $T>0$ is the length of a time interval,

$$\boldsymbol{w}=(w_1,\ldots,w_4)^{\mathrm{T}}=(\rho,\,\rho v_1,\,\rho v_2,\,E)^{\mathrm{T}} \quad (2)$$

is the so-called state vector and

$$\boldsymbol{f}_s(\boldsymbol{w}) \quad (3)$$
$$=(\rho v_s,\,\rho v_s v_1+\delta_{s1}p,\,\rho v_s v_2+\delta_{s2}p,\,(E+p)\,v_s)^{\mathrm{T}}$$

are the inviscid (Euler) fluxes of the quantity $\boldsymbol{w}$ in the directions x_s, $s=1,2$. We use the following notation: ρ – density, p – pressure, E – total energy, $\boldsymbol{v}=(v_1,v_2)$ – velocity, δ_{sk} – Kronecker symbol. The equation of state implies that

$$p=(\gamma-1)\,(E-\rho|\boldsymbol{v}|^2/2). \quad (4)$$

Here $\gamma>1$ is the Poisson adiabatic constant. The system (1) – (4) is *hyperbolic.* It is equipped with the initial condition

$$\boldsymbol{w}(\boldsymbol{x},0)=\boldsymbol{w}^0(x), \quad x\in\Omega, \quad (5)$$

and the boundary conditions, which are treated in Section 5. We define the matrix

$$\boldsymbol{P}(\boldsymbol{w},\boldsymbol{n}):=\sum_{s=1}^{2}\boldsymbol{A}_s(\boldsymbol{w})n_s, \quad (6)$$

where $\boldsymbol{n}=(n_1,n_2)\in I\!R^2,\ n_1^2+n_2^2=1$ and

$$\boldsymbol{A}_s(\boldsymbol{w})=\frac{D\boldsymbol{f}_s(\boldsymbol{w})}{D\boldsymbol{w}},\ s=1,2, \quad (7)$$

are the Jacobi matrices of the mappings $\boldsymbol{f}_s$. It is possible to show that $\boldsymbol{f}_s$, $s=1,2$, are homogeneous mappings of order one, which implies that

$$\boldsymbol{f}_s(\boldsymbol{w})=\boldsymbol{A}_s(\boldsymbol{w})\boldsymbol{w},\ s=1,2. \quad (8)$$

3 Discretization

Let Ω_h be a polygonal approximation of Ω. By $\mathcal{T}_h$ we denote a partition of Ω_h consisting of various types of convex elements $K_i\in\mathcal{T}_h,\ i\in I$ ($I\subset Z^+=\{0,1,2,\ldots\}$ is a suitable index set), e. g., triangles, quadrilaterals or in general convex polygons. (Let us note that in [4] it was shown that in the DGFEM also general nonconvex star-shaped polygonal elements can be used.) By Γ_{ij} we denote a common edge between two neighbouring elements K_i and K_j. The symbol $\boldsymbol{n}_{ij}=((n_{ij})_1,(n_{ij})_2)$ denotes the unit outer normal to ∂K_i on the side Γ_{ij}. Moreover, we set $s(i)=\{j\in I; K_j$ is a neighbour of $K_i\}$. The boundary $\partial\Omega_h$ is formed by a finite number of faces of elements K_i adjacent to $\partial\Omega_h$. We denote all these boundary faces by S_j, where $j\in I_b\subset Z^-=\{-1,-2,\ldots\}$. Now we set $\gamma(i)=\{j\in I_b; S_j$ is a face of $K_i\in\mathcal{T}_h\}$ and $\Gamma_{ij}=S_j$ for $K_i\in\mathcal{T}_h$ such that $S_j\subset\partial K_i,\ j\in I_b$. For K_i not containing any boundary face S_j we set $\gamma(i)=\emptyset$. Obviously, $s(i)\cap\gamma(i)=\emptyset$ for all $i\in I$. Now, if we write $S(i)=s(i)\cup\gamma(i)$, we have

$$\partial K_i=\bigcup_{j\in S(i)}\Gamma_{ij}, \quad \partial K_i\cap\partial\Omega_h=\bigcup_{j\in\gamma(i)}\Gamma_{ij}. \quad (9)$$

The approximate solution will be sought at each time instant t as an element of the finite-dimensional space

$$S_h=S^{r,-1}(\Omega_h,\mathcal{T}_h) \quad (10)$$
$$=\{v; v|_K\in P^r(K)\ \forall\, K\in\mathcal{T}_h\}^4,$$

where $r\geq 0$ is an integer and $P^r(K)$ denotes the space of all polynomials on K of degree $\leq r$. Functions $v\in S_h$ are in general discontinuous on interfaces Γ_{ij}. By $v|_{\Gamma_{ij}}$ and $v|_{\Gamma_{ji}}$ we denote the values of v on Γ_{ij} considered from the interior and the exterior of K_i, respectively.

In order to derive the discrete problem, we multiply (1) by a test function $\boldsymbol{\varphi}\in S_h$, integrate over any element $K_i,\ i\in I$, apply Green's theorem and sum over all $i\in I$. Then we approximate fluxes through the faces Γ_{ij} with the aid of a numerical flux $\boldsymbol{H}=\boldsymbol{H}(\boldsymbol{u},\boldsymbol{w},\boldsymbol{n})$ in the form

$$\int_{\Gamma_{ij}}\sum_{s=1}^{2}\boldsymbol{f}_s(\boldsymbol{w}(t))\,(n_{ij})_s\cdot\boldsymbol{\varphi}\,\mathrm{d}S \quad (11)$$
$$\approx\int_{\Gamma_{ij}}\boldsymbol{H}(\boldsymbol{w}_h(t)|_{\Gamma_{ij}},\boldsymbol{w}_h(t)|_{\Gamma_{ji}},\boldsymbol{n}_{ij})\cdot\boldsymbol{\varphi}\,\mathrm{d}S.$$

If we introduce the forms

$$(\boldsymbol{w}_h, \boldsymbol{\varphi}_h)_h = \int_{\Omega_h} \boldsymbol{w}_h \cdot \boldsymbol{\varphi}_h \,\mathrm{d}\boldsymbol{x}, \qquad (12)$$
$$\tilde{b}_h(\boldsymbol{w}_h, \boldsymbol{\varphi}_h) = \sigma_1 + \sigma_2,$$

where

$$\sigma_1 = -\sum_{K\in\mathcal{T}_h} \int_K \sum_{s=1}^{2} \boldsymbol{f}_s(\boldsymbol{w}_h) \cdot \frac{\partial \boldsymbol{\varphi}_h}{\partial x_s} \,\mathrm{d}\boldsymbol{x}, \qquad (13)$$

$$\sigma_2 = \sum_{K_i\in\mathcal{T}_h} \sum_{j\in S(i)} \int_{\Gamma_{ij}} \boldsymbol{H}(\boldsymbol{w}_h|_{\Gamma_{ij}}, \boldsymbol{w}_h|_{\Gamma_{ji}}, \boldsymbol{n}_{ij}) \cdot \boldsymbol{\varphi}_h \mathrm{d}S,$$

we can define an *approximate solution* of (1) as a function w_h satisfying the conditions

$$\text{a)} \quad \boldsymbol{w}_h \in C^1([0,T]; S_h), \qquad (14)$$
$$\text{b)} \quad \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{w}_h(t), \boldsymbol{\varphi}_h)_h + \tilde{b}_h(\boldsymbol{w}_h(t), \boldsymbol{\varphi}_h) = 0 \quad \forall\, \boldsymbol{\varphi}_h \in S_h \ \forall\, t \in (0,T),$$
$$\text{c)} \quad \boldsymbol{w}_h(0) = \Pi_h \boldsymbol{w}^0,$$

where $\Pi_h \boldsymbol{w}^0$ is the L^2-projection of $\boldsymbol{w}^0$ from the initial condition (5) on the space S_h. If we set $r = 0$, then we obviously obtain the finite volume method.

4 Time discretization

Relation (14), b) represents a system of ordinary differential equations which can be solved by a suitable numerical method. Usually, *Runge-Kutta schemes* are applied. However, they are conditionally stable and the time step is strongly limited by the CFL-stability condition. Another possibility is to use the fully implicit *backward Euler method*, but it leads to a large system of highly nonlinear algebraic equations whose numerical solution is rather complicated. Our aim is to obtain a higher order unconditionally stable scheme, which would require the solution of a linear system on each time level. This is carried out with the aid of a suitable partial linearization of the form $\tilde{b}_h$. In what follows, we consider a partition $0 = t_0 < t_1 < t_2 \ldots$ of the time interval $(0,T)$ and set $\tau_k = t_{k+1} - t_k$. We use the notation $\boldsymbol{w}_h^k$ for the approximation of $\boldsymbol{w}_h(t_k)$.

In [3] we described a new DG semi-implicit technique which is suitable for an efficient solution of inviscid stationary as well as nonstationary compressible flow. This technique is based on a linearization of the forms σ_1 and σ_2, defined by (13) and using the Vijayasundaram numerical flux. In this way we obtain the form

$$b_h(\boldsymbol{w}_h^k, \boldsymbol{w}_h^{k+1}, \boldsymbol{\varphi}_h) \qquad (15)$$
$$= \sum_{K\in\mathcal{T}_h} \int_K \sum_{s=1}^{2} \boldsymbol{A}_s(\boldsymbol{w}_h^k(\boldsymbol{x}))\boldsymbol{w}_h^{k+1}(\boldsymbol{x}) \cdot \frac{\partial \boldsymbol{\varphi}_h(\boldsymbol{x})}{\partial x_s} \,\mathrm{d}\boldsymbol{x}$$
$$+ \sum_{K_i\in\mathcal{T}_h} \sum_{j\in S(i)} \int_{\Gamma_{ij}} \Big[\boldsymbol{P}^+\left(\langle \boldsymbol{w}_h^k\rangle_{ij}, \boldsymbol{n}_{ij}\right) \boldsymbol{w}_h^{k+1}|_{\Gamma_{ij}}$$
$$+ \boldsymbol{P}^-\left(\langle \boldsymbol{w}_h^k\rangle_{ij}, \boldsymbol{n}_{ij}\right) \boldsymbol{w}_h^{k+1}|_{\Gamma_{ji}}\Big] \cdot \boldsymbol{\varphi}_h \mathrm{d}S,$$

which is linear with respect to the second and third variable. We use the notation $\langle \boldsymbol{w}_h^k\rangle_{ij} = (\boldsymbol{w}_h|_{\Gamma_{ij}} + \boldsymbol{w}_h|_{\Gamma_{ji}})/2$. Further, $\boldsymbol{P}^\pm = \boldsymbol{P}^\pm(\boldsymbol{w}, \boldsymbol{n})$ represents positive/negative part of the matrix $\boldsymbol{P}$ defined on the basis of its diagonalization (see, e.g. [7], Section 3.1):

$$\boldsymbol{P} = \boldsymbol{T}\boldsymbol{D}\boldsymbol{T}^{-1}, \ \boldsymbol{D} = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_4), \qquad (16)$$

where $\lambda_1, \ldots, \lambda_4$ are the eigenvalues of $\boldsymbol{P}$. Then we set

$$\boldsymbol{D}^\pm = \mathrm{diag}\,(\lambda_1^\pm, \ldots, \lambda_4^\pm), \qquad (17)$$
$$\boldsymbol{P}^\pm = \boldsymbol{T}\boldsymbol{D}^\pm\boldsymbol{T}^{-1},$$

where $\lambda^+ = \max\{a, 0\}$ and $\lambda^- = \min\{a, 0\}$.

On the basis of the above considerations we obtain the following semi-implicit scheme: For each $k \geq 0$ find $\boldsymbol{w}_h^{k+1}$ such that

$$\text{a)} \quad \boldsymbol{w}_h^{k+1} \in \boldsymbol{S}_h, \qquad (18)$$
$$\text{b)} \quad \left(\frac{\boldsymbol{w}_h^{k+1} - \boldsymbol{w}_h^k}{\tau_k}, \boldsymbol{\varphi}_h\right)_h + b_h(\boldsymbol{w}_h^k, \boldsymbol{w}_h^{k+1}, \boldsymbol{\varphi}_h) = 0 \quad \forall\, \boldsymbol{\varphi}_h \in \boldsymbol{S}_h, \ k = 0, 1, \ldots,$$
$$\text{c)} \quad \boldsymbol{w}_h^0 = \Pi_h \boldsymbol{w}^0.$$

This is a first order accurate scheme in time. In [3] also a two step second order time discretization was proposed. The linear algebraic system equivalent to (18), b) is solved by the GMRES method with a block diagonal preconditioning. In order to guarantee the stability of the scheme, we use the CFL condition

$$\tau_k \max_{K_i\in\mathcal{T}_h} \frac{1}{|K_i|}\left(\max_{j\in S(i)} |\Gamma_{ij}| \lambda^{\max}_{\boldsymbol{P}(\boldsymbol{w}_h^k|_{\Gamma_{ij}}, \boldsymbol{n}_{ij})}\right) \leq \mathrm{CFL}, \qquad (19)$$

where $|K_i|$ denotes the area of K_i, $|\Gamma_{ij}|$ the length of the edge Γ_{ij}, CFL a given constant and

$\lambda^{\max}_{\boldsymbol{P}(\boldsymbol{w}_h^k|_{\Gamma_{ij}},\boldsymbol{n}_{ij})}$ is the maximal eigenvalue of the matrix $\boldsymbol{P}(\boldsymbol{w}_h^k|_{\Gamma_{ij}},\boldsymbol{n}_{ij})$ defined by (6), where the maximum is taken over Γ_{ij}. Numerical experiments show that the CFL number can be practically unlimited.

In order to obtain an accurate, physically admissible solution, it is necessary to add two further ingredients to the computational process:

In the case of curved boundaries, it is necessary to use superparametric elements (see [1] or [2]).

For the flow with internal or boundary layers (shock waves, contact discontinuities, boundary layers) it is necessary to avoid the Gibbs phenomenon manifested by spurious overshoots and undershoots in computed quantities. In [5] we proposed a method for avoiding this phenomenon using the limiting of order of accuracy of the scheme in a vicinity of discontinuities and steep gradients. Here we do not need its application, because this paper is concerned with low Mach number flows only.

5 Boundary conditions

If $\Gamma_{ij} \subset \partial\Omega_h$, i.e. $j \in \gamma(i)$, it is necessary to specify the boundary state $\boldsymbol{w}|_{\Gamma_{ji}}$ appearing in the numerical flux $\boldsymbol{H}$ in the definition of the inviscid form b_h. The appropriate treatment of boundary conditions plays a crucial role in the solution of low Mach number flows.

On a fixed impermeable wall we employ a standard approach using the condition $\boldsymbol{v}\cdot\boldsymbol{n} = 0$ and extrapolating the pressure. On the inlet and outlet it is necessary to use nonreflecting boundary conditions transparent for acoustic effects coming from inside of Ω. Therefore, *characteristics based* boundary conditions are used.

Using the rotational invariance, we transform the Euler equations to the coordinates $\tilde{x}_1$, parallel with the normal direction $\boldsymbol{n}$ to the boundary, and $\tilde{x}_2$, tangential to the boundary, neglect the derivative with respect to $\tilde{x}_2$ and linearize the system around the state $\boldsymbol{q}_{ij} = \boldsymbol{Q}(\boldsymbol{n}_{ij})\boldsymbol{w}|_{\Gamma_{ij}}$, where

$$\boldsymbol{Q}(\boldsymbol{n}_{ij}) = \begin{pmatrix} 1, & 0, & 0, & 0 \\ 0, & (n_{ij})_1, & (n_{ij})_2, & 0 \\ 0, & -(n_{ij})_2, & (n_{ij})_1, & 0 \\ 0, & 0, & 0, & 1 \end{pmatrix} \qquad (20)$$

is the rotational matrix. Then we obtain the linear system

$$\frac{\partial \boldsymbol{q}}{\partial t} + \boldsymbol{A}_1(\boldsymbol{q}_{ij})\frac{\partial \boldsymbol{q}}{\partial \tilde{x}_1} = 0, \qquad (21)$$

for the transformed vector-valued function $\boldsymbol{q} = \boldsymbol{Q}(\boldsymbol{n}_{ij})\boldsymbol{w}$, considered in the set $(-\infty,0)\times(0,\infty)$ and equipped with the initial and boundary conditions

$$\boldsymbol{q}(\tilde{x}_1,0) = \boldsymbol{q}_{ij}, \quad \tilde{x}_1 \in (-\infty,0), \qquad (22)$$
$$\boldsymbol{q}(0,t) = \boldsymbol{q}_{ji}, \quad t > 0.$$

The goal is to choose $\boldsymbol{q}_{ji}$ in such a way that this initial-boundary value problem is well posed, i.e. has a unique solution. The method of characteristics leads to the following process:

Let us put $\boldsymbol{q}^*_{ji} = \boldsymbol{Q}(\boldsymbol{n}_{ij})\boldsymbol{w}^*_{ji}$, where $\boldsymbol{w}^*_{ji}$ is a prescribed boundary state at the inlet or outlet. We calculate the eigenvectors $\boldsymbol{r}_s$ corresponding to the eigenvalues λ_s, $s = 1,\dots,4$, of the matrix $\boldsymbol{A}_1(\boldsymbol{q}_{ij})$, arrange them as columns in the matrix $\boldsymbol{T}$ and calculate $\boldsymbol{T}^{-1}$ (explicit formulae can be found in [7], Section 3.1). Now we set

$$\boldsymbol{\alpha} = \boldsymbol{T}^{-1}\boldsymbol{q}_{ij}, \quad \boldsymbol{\beta} = \boldsymbol{T}^{-1}\boldsymbol{q}^*_{ji}. \qquad (23)$$

and define the state $\boldsymbol{q}_{ji}$ by the relations

$$\boldsymbol{q}_{ji} := \sum_{s=1}^{4} \gamma_s \boldsymbol{r}_s, \quad \gamma_s = \begin{cases} \alpha_s, & \lambda_s \ge 0, \\ \beta_s, & \lambda_s < 0. \end{cases} \qquad (24)$$

Finally, the sought boundary state $\boldsymbol{w}|_{\Gamma_{ji}}$ is defined as

$$\boldsymbol{w}|_{\Gamma_{ji}} = \boldsymbol{w}_{ji} = \boldsymbol{Q}^{-1}(\boldsymbol{n}_{ij})\boldsymbol{q}_{ji}. \qquad (25)$$

6 Numerical example

In [3], several examples of transonic flow calculations are presented. They prove the accuracy and efficiency of our method for the solution of high Mach number flows.

In order to show the robustness of the described technique with respect to low Mach numbers, we present interesting results obtained by the semi-implicit scheme (18) for stationary inviscid flow past a circular cylinder with the far field velocity parallel to the axis x_1 and Mach number $M_\infty = 10^{-4}$. The problem was solved in a computational domain in the form of a square with sides of the length equal to 20 diameters of the cylinder. We show here details of the flow in the vicinity of the cylinder. Figure 1 shows isolines of the absolute value of the velocity for the compressible flow computed by scheme (18) with $r = 2$, compared with the exact solution of incompressible flow (computed by the method of

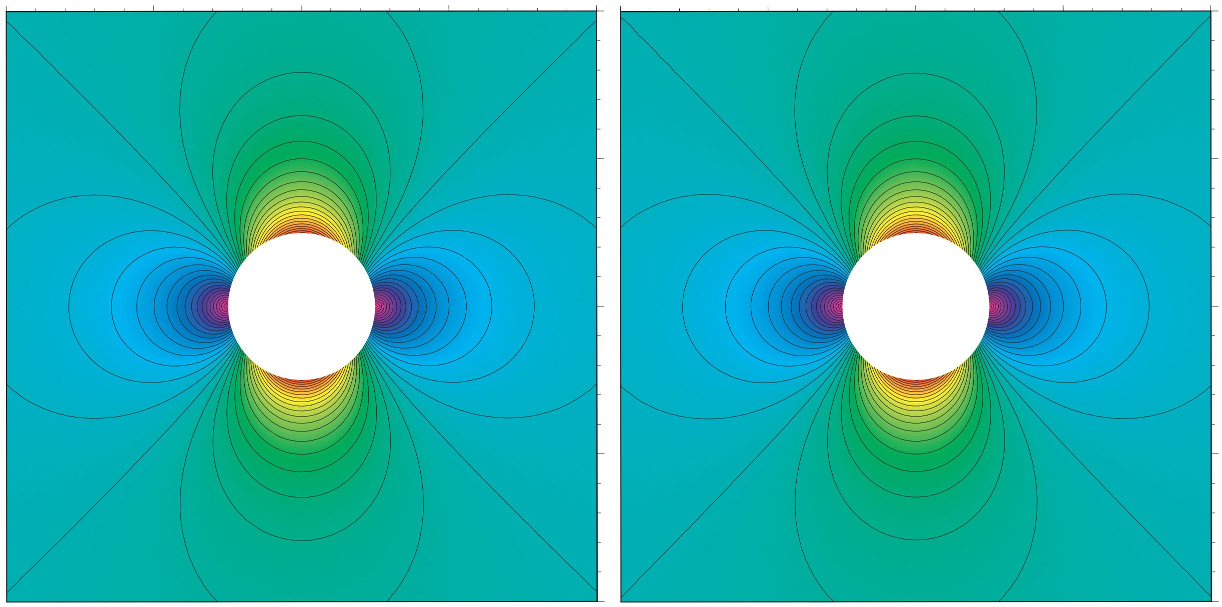

Figure 1: Velocity isolines for the approximate solution of compressible flow (left) and for the exact solution of incompressible flow (right)

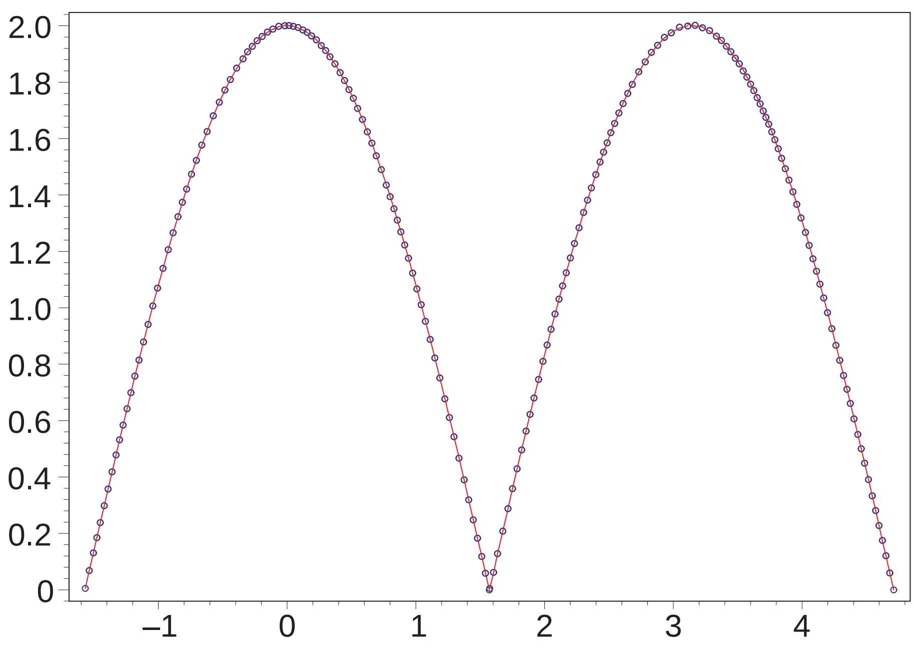

Figure 2: Velocity distribution along the cylinder (full line – compressible flow, dotted line – icompressible flow)

complex functions – see [6], Section 2.2.35). In Figure 2, the distribution of the absolute value of the velocity along the cylinder (related to the magnitude of the far field velocity) is shown. We see that the compressible and incompressible velocity distributions are identical. The computational process started with CFL = 38, which was gradually increased up to 2000. Thus, the method is practically *unconditionally stable*. The steady state was reached after 300 time steps (when the maximum norm of the approximated time derivative was less than 10^{-8}). From the figures we see that the obtained solution is symmetric and the scheme does not produce any wake behind the cylinder. The computed flow behaves nearly as incompressible. The difference of the maximal and minimal values of the approximation ρ_h of the density, $\rho_{h\max} - \rho_{h\min} = 2.3 \cdot 10^{-8}$ and $\max_{i \in I} |\nabla \rho_h|_{K_i}| < 1.99 \cdot 10^{-6}$ which means that the density is practically constant. The computed density variation corresponds to the theoretical estimate following from known formulae (see, e.g. [9], Section 23).

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