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Optimization of Multiple Thin Thermal Insulation Layers

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Abstract: - Given prescribed quantities of two or more thermal insulation materials, how should they be distributed around a body so as to minimize the rate of loss of heat from that body? The application of the calculus of variations enables us to answer this question. The governing optimal coupled boundary-value problem posed over the domains occupied by the two materials has been formulated previously by the author and solved for the special case of a spherical body. In this paper the optimal boundary-value problem is posed for the general two-dimensional case where several thin layers of material are to be distributed around an arbitrary infinite prismatic body, whose surface has smoothly varying curvature. A regular perturbation procedure is applied to obtain the solution. The interesting finding emerges, that there is no unique optimal solution, but a family of solutions satisfying the optimality conditions.

Key-Words: - Optimization, Thermal Insulation, Heat Conduction, Calculus of Variations, Variational Principle, Multiple Layers, Global Warming

1 Introduction

The problem of optimising the shape of a body to maximise or minimise some property associated with it has interested mathematicians and engineers for centuries. Perhaps the most fundamental example of this type of problem is that of minimising the surface area of a body while keeping its volume fixed, which has the well-known solution, the sphere. This is an example of the class of problem known as isoperimetric. The author [1] has provided a short review of examples of this class.

This paper is concerned with a particular example of this class, namely the minimisation of the heat loss from a body of arbitrary shape by the variation of the shape of the multiple insulation layers around it. The minimisation is subject to the constraints that the volume, or equivalently the mass, of each layer is prescribed, and that the temperatures on the surface of the body, and on the outside of the outer layer, are also prescribed. Thus there is a limited quantity of insulation and the requirement is to distribute it optimally to reduce heat loss to a minimum, given the prescribed temperature drop across the insulation layers. This temperature difference may vary around the body.

The mathematics of the derivation of the governing boundary value problem has been presented in the preceding sister paper by the author [1]. In that paper he showed that, in addition to the governing heat conduction equations holding in each of the domains occupied by each body, the associated Dirichlet conditions and the continuity of the temperature and heat flux at the interfaces between adjacent insulation layers, three optimal boundary conditions hold, together with additional field equations in two 'adjoint' field variables. These new field variables were introduced to enable the elimination of awkward variations in derivatives of the temperature. The treatment bore similarity to the treatment by the author [2] of a single layer with a mixed boundary condition representing Newton's Law of Cooling.

We do not repeat the derivation of the variational minimisation principle here, but simply state it. Our concern here is with the detailed derivation of the optimal solution for the important general case of an infinite prism of arbitrary cross-section, with smoothly varying curvature of the boundary of the cross-section within its own plane. We address the situation where the thicknesses of the insulation layers are small compared with the associated radius of curvature of the body.

We apply a regular perturbation method, first used on this type of optimisation problem by Banichuk [3]. We solve to the first order and obtain a family of optimal solutions. It is shown that they can improve significantly upon the standard practice of applying constant thickness layers.

We expect the method to find application where mass constraints and extreme temperature variations dominate, e.g. in insulation of spacecraft components. Moreover, the use of resources in the most efficient way is highly pertinent as one means of attack on the global warming issue.

2 Problem Formulation

The coupled-domain optimal boundary value problem is shown in Figure 1 for the case of two layers of insulation. Extra layers can be treated analogously, as described by the author [1]. It addresses the heat flow in the two layers of insulation represented by domains Ω_1, Ω_2 surrounding a body B respectively. As in [1], T_1, T_2 are the temperatures in Ω_1, Ω_2, respectively. The inner surface of Ω_1 is S_0, the outer surface of Ω_1 (and hence the inner surface of Ω_2) is S_1, and the outer surface of Ω_2 is S_2. The temperatures on these surfaces S_0, S_1, S_2 are T_0^*, T_1^*, T_2^* respectively, where T_0^* and T_2^* are known, but the interface temperature T_1^* is not.

In Ω_1 the heat conduction equation is

$$\nabla^2 T_1 = 0 \ . \tag{1}$$

Similarly in Ω_2

$$\nabla^2 T_2 = 0 \ . \tag{2}$$

The boundary conditions are:

$$T_1 = T_0^* \text{ on } S_0, \tag{3}$$

$$T_1 = T_2 = T_1^* \text{ on } S_1, \tag{4}$$

and

$$T_2 = T_2^* \text{ on } S_2. \tag{5}$$

Continuity of the heat flux yields

$$-\kappa_1 \underline{n}_1 \cdot \nabla T_1 = \kappa_2 \underline{n}_2 \cdot \nabla T_2 \text{ on } S_1, \tag{6}$$

where κ_1 and κ_2 are the thermal conductivities.

The total rate of heat loss from the body B through the surface S_0 is given by

$$\int_{S_0} \kappa_1 \underline{n}_1 \cdot \nabla T_1 \, dS \ . \tag{7}$$

The author [1] introduced two further new variables v_1 and v_2, defined in Ω_1 and in Ω_2, with the only constraints imposed upon these functions being that they be twice differentiable and that

$$v_1 = v_2 \text{ on } S_1. \tag{8}$$

The heat loss was then written as a functional

$$I[T_1, T_2, v_1, v_2; \Omega_1, \Omega_2] = \int_{S_0} \kappa_1 \underline{n}_1 \cdot \nabla T_1 \, dS + \int_{S_1} \{v_1(-\kappa_1 \underline{n}_1 \cdot \nabla T_1) + v_2(-\kappa_2 \underline{n}_2 \cdot \nabla T_2)\} dS. \tag{9}$$

The constraint (8) and boundary condition (6) ensure that this functional always equals the rate of heat loss (7). In [1] it is the functional (9) that is made stationary to find the minimum heat loss from S_0.

The extra Euler equations

$$\nabla^2 v_1 = 0 \text{ in } \Omega_1, \tag{10}$$

and

$$\nabla^2 v_2 = 0 \text{ in } \Omega_2 \tag{11}$$

are found to apply. Since the role of the variables v_1 and v_2 was to act as Lagrange multipliers on S_1, we imposed the boundary conditions

$$v_1 = -1 \text{ on } S_0, \tag{12}$$

$$v_2 = 0 \text{ on } S_2. \tag{13}$$

Consideration of the variations of v_1 and v_2 satisfying constraint (8) yields the optimality condition

$$\kappa_1 \frac{\partial v_1}{\partial n_1} = -\kappa_2 \frac{\partial v_2}{\partial n_2} \text{ on } S_1. \tag{14}$$

Further optimal boundary conditions result from the variational principle:

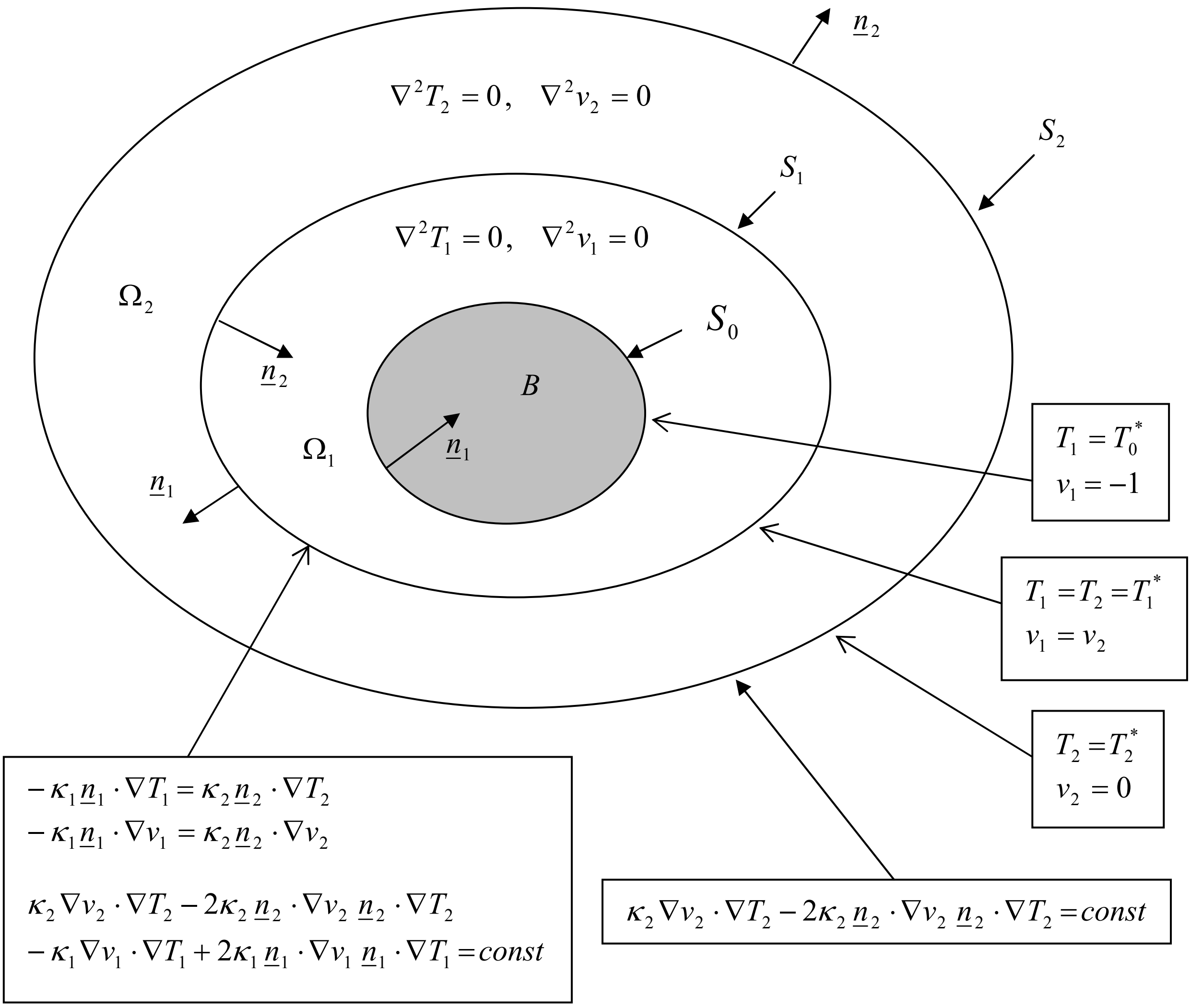

Fig.1. The body B, the two surrounding insulation layers occupying domains Ω_1, Ω_2 and the governing coupled boundary-value problems in the temperatures and adjoint variables.

$$\kappa_2 \nabla v_2 \cdot \nabla T_2 - 2\kappa_2 \frac{\partial v_2}{\partial n_2}\frac{\partial T_2}{\partial n_2} - \kappa_1 \nabla v_1 \cdot \nabla T_1 + 2\kappa_1 \frac{\partial v_1}{\partial n_1}\frac{\partial T_1}{\partial n_1} = const, \quad (15)$$

holding on S_1, and

$$\kappa_2 \nabla v_2 \cdot \nabla T_2 - 2\kappa_2 \frac{\partial v_2}{\partial n_2}\frac{\partial T_2}{\partial n_2} = const, \quad (16)$$

holding on S_2.

All of the equations and boundary conditions for the optimal boundary–value problem are now stated and we may proceed to seek their solution.

3 Solution for Infinite Prism

Consider now the case of an infinite prism, in which Fig. 1 now represents a right cross-section of the infinitely long body. This solution will provide a good approximation to the temperature field and heat flow for the mid regions of long bodies where the end-effects will be negligible. We confine our attention as yet to the case where there is smoothly

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varying curvature of the boundary of the cross-section within its own plane. We furthermore address the situation where the thicknesses of the insulation layers are small compared with the associated radius of curvature of the body, which is usually a desirable characteristic of insulation layers.

We now apply an extension of the regular perturbation method of Banichuk [3]. We introduce intrinsic co-ordinates (s,t) on S_1, where s is the arc length around the boundary of the cross-section, measured in the anti-clockwise sense and t is the orthogonal co-ordinate taken outwards from S_1.

If we introduce the small parameter ε given by the ratio of the cross-sectional area A of the insulation layers to the square of the perimeter of S_1, denoted by L, so that

$$\varepsilon = \frac{A}{L^2}, \tag{17}$$

we may seek a regular perturbation solution to the boundary-value problem established by eqs. (1)-(16) and displayed in Fig. 1. Denote the zero-th order values of each variable by a subscript '0', and let the surfaces S_1 and S_2 be given by

$$t = h_{10}(s), \tag{18}$$

and

$$t = h_{10}(s) + h_{20}(s), \tag{19}$$

respectively. We may then write the zero-th order problem as follows, relation by relation, denoting derivatives with respect to s and t by subscripts of the respective variable. Addressing the domain Ω_1 first, Eq. (1) becomes

$$T_{10tt} = 0, \qquad 0 \le t \le h_{10}, \tag{20}$$

Let $T_0^* = T_0(s)$. Then condition (3) becomes

$$T_{10} = T_0(s), \quad \text{on } t = 0. \tag{21}$$

Condition (4) is simply

$$T_{10} = T_{20}, \quad \text{on } t = h_{10}. \tag{22}$$

The interface condition (6) reduces to

$$\kappa_1 T_{10t} = \kappa_2 T_{20t}, \quad \text{on } t = h_{10}. \tag{23}$$

Condition (8) in the adjoint variables is now

$$v_{10} = v_{20}, \quad \text{on } t = h_{10}. \tag{24}$$

Equation (10) becomes

$$v_{10tt} = 0, \qquad 0 \le t \le h_{10}. \tag{25}$$

The imposed boundary condition (12) is

$$v_{10} = -1, \quad \text{on } t = 0. \tag{26}$$

The optimality condition (14) is

$$\kappa_1 v_{10t} = \kappa_2 v_{20t}, \quad \text{on } t = h_{10}. \tag{27}$$

It may be shown that the optimality condition (15) reduces to

$$(T_{10t} + T_{20t})v_{20t} = \lambda_{10}, \quad \text{on } t = h_{10}(s), \tag{28}$$

where λ_{10} is a constant to be found.

Now consider the domain Ω_2. Eq. (2) becomes

$$T_{20tt} = 0, \qquad h_{10} \le t \le h_{10} + h_{20}. \tag{29}$$

Condition (5) becomes

$$T_{20} = T_2^*, \quad \text{on } t = h_{10} + h_{20}. \tag{30}$$

Equation (11) becomes

$$v_{20tt} = 0, \qquad h_{10} \le t \le h_{10} + h_{20}. \tag{31}$$

The boundary condition (13) becomes

$$v_{20} = 0, \quad \text{on } t = h_{10} + h_{20}. \tag{32}$$

Condition (16) reduces to

$$T_{20t} v_{20t} = \lambda_{20}, \qquad \text{on } t = h_{10} + h_{20}, \tag{33}$$

where λ_{20} is a constant to be determined. Finally, we introduce the cross-sectional areas of the inner and outer layers as a_1 and a_2, so that

$$\int_0^L h_{10}\, ds = a_1 \tag{34}$$

and

$$\int_0^L h_{20}\, ds = a_2 \ . \tag{35}$$

We may now solve eqs. (17)-(35) to obtain the zero-th-order solution to the pair of coupled boundary value problems given by eqs. (1)-(16) as

$$T_{10} = \frac{\kappa_2\left(T_2^* - T_0(s)\right)t}{(\kappa_2 h_{10} + \kappa_1 h_{20})} + T_0(s)\,, \tag{36}$$

$$T_{20} = \frac{\kappa_1\left(T_2^* - T_0(s)\right)\left(t - h_{10} - h_{20}\right)}{\left(\kappa_2 h_{10} + \kappa_1 h_{20}\right)} + T_2^*\,, \tag{37}$$

$$v_{10} = \frac{\kappa_2 t}{\left(\kappa_2 h_{10} + \kappa_1 h_{20}\right)} - 1\,, \tag{38}$$

$$v_{20} = \frac{\kappa_1\left(t - h_{10} - h_{20}\right)}{\left(\kappa_2 h_{10} + \kappa_1 h_{20}\right)}\,, \tag{39}$$

$$\lambda_{10} = -\frac{\kappa_2\left(\kappa_1 + \kappa_2\right)\left\{\int_0^L \left(T_0(s) - T_2^*\right)^{\frac{1}{2}} ds\right\}^2}{\left(\kappa_2 a_1 + \kappa_1 a_2\right)^2} \tag{40}$$

and

$$\lambda_{20} = -\frac{\kappa_1^2\left\{\int_0^L \left(T_0(s) - T_2^*\right)^{\frac{1}{2}} ds\right\}^2}{\left(\kappa_2 a_1 + \kappa_1 a_2\right)^2}\,. \tag{41}$$

It is then straightforward to show that the local heat flux is given by

$$\frac{\kappa_1\kappa_2\left(T_0(s) - T_2^*\right)^{\frac{1}{2}}\int_0^L \left(T_0(s) - T_2^*\right)^{\frac{1}{2}} ds}{\left(\kappa_2 a_1 + \kappa_1 a_2\right)}\,. \tag{42}$$

Thus the optimal, minimal, total rate of heat loss from S_0 is given by

$$\frac{\kappa_1\kappa_2\left\{\int_0^L \left(T_0(s) - T_2^*\right)^{\frac{1}{2}} ds\right\}^2}{\left(\kappa_2 a_1 + \kappa_1 a_2\right)}\,. \tag{43}$$

4 Discussion

It is an interesting aspect of the solution that individual explicit expressions for h_{10} and h_{20} do not appear. They are, however, related by the relationship

$$\kappa_2 h_{10} + \kappa_1 h_{20} = \kappa_1\left(\frac{T_2^* - T_0(s)}{\lambda_{20}}\right)^{\frac{1}{2}} \tag{44}$$

and so a family of optimal solutions is possible so long as Eqs. (34), (35) and (44) are satisfied. One can vary the thicknesses of the layers in any way so long as these equations are satisfied and still deliver the optimal lowest heat loss.

It is also of interest to compare with the solution of constant thickness layers, the common practice in application of insulation. In this case,

$$h_{10} = \frac{a_1}{L}\,,\ h_{20} = \frac{a_2}{L}\,. \tag{45}$$

The solutions for T_{10} and T_{20} are in fact identical to those given by eqs. (36) and (37), but the corresponding total heat loss is now given by

$$\frac{\kappa_1\kappa_2 L\int_0^L \left(T_0(s) - T_2^*\right) ds}{\left(\kappa_2 a_1 + \kappa_1 a_2\right)}\,. \tag{46}$$

Following Banichuk's [3] similar treatment for torsional rigidity, Schwarz's inequality can be applied to the expressions (43) and (46) to show that the optimal solution indeed gives less heat loss than the constant thicknesses solution. We show an example of the saving made in the next section.

At present we have limited the analysis for the multiple layer case to a Dirichlet condition on the outer surface S_2, but, as discussed by Curtis [1], one may consider radiation or mixed boundary conditions such as result from Newton's Law of Cooling on the outer surface. Other problems of interest would include the relaxation of the constraints of thin layers and a smoothly curving body. Such extensions lie beyond the scope of this paper.

5 An Example

Let the total temperature difference across the double insulation layer be:

$$T_0(s) - T_2^* = \theta s^2 (L-s)^2 L^{-4}. \qquad (47)$$

It is straightforward to evaluate the following integrals:

$$\int_0^L \left(T_0(s) - T_2^*\right)^{\frac{1}{2}} ds = \frac{\theta^{\frac{1}{2}} L}{6}, \qquad (48)$$

$$\int_0^L \left(T_0(s) - T_2^*\right) ds = \frac{\theta L}{30}. \qquad (49)$$

Expressions (43) and (46) then reduce respectively to

$$\frac{\kappa_1 \kappa_2 \theta L^2}{36\left(\kappa_2 a_1 + \kappa_1 a_2\right)}, \quad \frac{\kappa_1 \kappa_2 \theta L^2}{30\left(\kappa_2 a_1 + \kappa_1 a_2\right)}, \qquad (50)$$

showing that for the temperature difference as defined by eq. (47) the optimal value is lower than the constant thicknesses solution by 16.67%.

6 Conclusions

The problem of deploying a double thin layer of insulation around a prismatic body of smoothly curved cross-sectional shape in such a way as to minimize heat loss from the body, subject to the constraints of keeping the mass of each layer constant has been addressed. The problem has first been posed mathematically and then solved by a regular perturbation procedure. The interesting finding has emerged, that there is no unique optimal solution, but rather a family of possible solutions satisfying the fundamental boundary-value problem and the optimality conditions. A proof that the optimal value of the heat loss lies below the traditionally applied and practical constant thicknesses solution has been indicated, based on Schwarz's inequality.

A simple example considering a smoothly varying temperature field has been solved explicitly and a saving of 16.67% results by adopting an optimal solution as compared with the constant thicknesses solution. The value of the saving will in general be dependent upon the temperature difference between the body and the outer layer. The use of the method is most likely to find application in situations where mass is a critical issue, such as in spacecraft, aerospace or possibly automobile engineering. However, energy saving is becoming ever more important in the face of the global warming issue and the method or refinements thereof may find applications in this connection.

The constraints imposed in this work, of thin layers and smooth curvature, may need to be relaxed for some future applications. It is expected that the requirement to address such situations numerically, rather than analytically as here, would then arise.

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