Proceedings of the 3rd IASME/WSEAS Int. Conf. on FLUID DYNAMICS & AERODYNAMICS, Corfu, Greece, August 20-22, 2005 (pp109-112)

Quasi–Lipschitz Conditions in Vorticity Transport

REIMUND RAUTMANN
Institut für Mathematik der Universität Paderborn
D 33095 Paderborn
GERMANY
rautmann@upb.de

Abstract: - Since vorticity transport is becoming a more and more important tool in recent numerical approaches to fluid flow problems, there is some interest in the study of approximations to its basic equation. In the joint paper [18] with V. Solonnikov, using quasi–Lipschitz conditions, we have shown that Helmholtz's vorticity transport equation with partial discretization has a unique classical solution in smoothly bounded 3-dimensional domains on each bounded interval of time. This solution depends continuously on its initial value and, in addition, fulfils a discretized form of Cauchy's vorticity equation.

Key-Words: - Helmholtz equation, vorticity

1 Introduction

The use of vorticity transport in mathematical models of incompressible flow problems is taking its advantage from the regularization being implied in the reconstruction of the flow velocity v from its vorticity $w = \operatorname{rot} v$: We get v from the boundary value problem

$$\operatorname{rot} v = w, \quad \operatorname{div} v = 0 \qquad \text{in } \Omega, \tag{1}$$

$$v \cdot N = 0 \qquad \text{on } \partial\Omega \tag{2}$$

in a bounded simply connected domain $\Omega \subset \mathbb{R}^3$ with boundary $S = \partial\Omega \in C^{2+\alpha}$, $\alpha \in (0,1)$; $N(x)$ denotes the exterior normal to S at the point x. It is well known from the vector potential theory that problem (1), (2) admits a unique solution

$$v = Kw \tag{3}$$

for arbitrary divergence free $w \in C^0(\overline{\Omega})$ satisfying the compatibility conditions

$$\int_{S_k} w \cdot N \, ds = 0, \quad k = 0, \ldots, h, \tag{4}$$

where S_k are the connected components of S (S_0 is the exterior and $S_1, \ldots S_h$ are the interior boundaries), and the operator K (being of Biot–Savart type) can be defined also in the case of multi–connected domains, c.p. [17] and the citations there.

The important question, wether we will arrive at a fluid flow having unique particles' pathes may be settled by requiring a spatial Lipschitz condition for the velocity field v from (1), (2). However, as shown in potential theory [4], this would require Hölder–continuity of w with respect to its spatial coordinates. A more natural picture would be given by bounds only for the sup-norm $|w(t,\cdot)|_0 = \sup_{x\in\Omega} |w(t,x)|$ of $w(t,\cdot)$. This leads to sup-norm bounds for $v(t,\cdot) = Kw(t,\cdot)$, but then, instead of a Lipschitz condition, only a **quasi-Lipschitz condition** can be established for v: There holds

$$[v(t,\cdot)]_\ell = \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|v(t,y) - v(t,x)|}{\ell(|y-x|)} \leq c \cdot \sup_{x\in\Omega} |w(t,x)| \tag{5}$$

with

$$\ell(r) = \begin{cases} 0, & r = 0, \\ -r \ln r, & r \in (0, e^{-1}), \\ r, & r \geq e^{-1} \end{cases} \tag{6}$$

the constant c being independent of w, [17, Theorem 1.4].

The basic description of the transport of vorticity $w = \operatorname{rot} v$ is given by the initial value problem of Helmholtz's equation

$$\frac{\partial}{\partial t} w + v \cdot \nabla w = w \cdot \nabla v, \quad 0 \leq t, \ x \in \overline{\Omega} \tag{7}$$

$$w(0,\cdot) = w_0, \qquad t = 0, \ x \in \overline{\Omega}, \tag{8}$$

which formally follows by taking "rot" in Euler's hydrodynamical equation (in the case of outer forces having a potential). In a 2-dimensional flow parallel to the (x,y)-plane, the right hand side in (7) becoming zero, a maximum principle holds for w, or, if the prescribed outer force does not have a potential, at least an upper bound for $|w(t,\cdot)|_0$ would be available.

In this way, decisively using quasi–Lipschitz bounds, Kato [6], Hölder [5], Wolibner [27], Yudowich [28] have proved global existence of unique classical solutions to Euler's equation in 2 space dimensions. For the many approaches to this open problem in 3 space dimensions consult [11], [12], [3] and the citations there.

In view of the well-known difficulties of 3-dimensional flows, it seems to be remarkable that a partial discretization in Helmholtz's vorticity transport equation (7) leads to initial value problems allowing unique classical solutions on each bounded interval of time.

2 Helmholtz's vorticity transport equation with partial discretization

In the following we consider domains $\Omega \subset \mathbb{R}^3$ which have the topological type of a ball $\mathcal{B}$ with m solid handles, $m \geq 0$, inside of $\mathcal{B}$ a number h of smaller balls $\overline{\Omega}_k$ being cut out. Such a domain is characterized (a) by the presence of a homotopy basis of m simply closed smooth curves $l_i \subset \Omega$ which inside Ω neither can be continuously deformed into each other nor into a single point, and (b) by the fact that making m cuts along smooth surfaces $\Sigma_i \subset \overline{\Omega}$ we can transform the domain Ω into a simply connected one. In each such $m+1$-times connected domain Ω with $C^{2+\alpha}$-boundary $\partial\Omega = S$ there exist precisely m linearly independent "Neumann vector fields" $u_i \in C^{1+\alpha}(\overline{\Omega})$ satisfying the conditions

$$\begin{aligned} \operatorname{rot} u_i &= 0, \quad \operatorname{div} u_i = 0 \quad \text{in } \Omega, \\ u_i \cdot N &= 0 \qquad\qquad\qquad\quad \text{on } S, \end{aligned}$$

u_i having the fluxes

$$\int_{\Sigma_k} u_i \cdot N \, ds = \delta_{ik}$$

across Σ_k, or the circulations

$$\int_{l_k} u_i \cdot dl = \delta_{ik} \, ,$$

$i, k = 1, \cdots, m$, respectively. Here $N(x)$ denotes a unit normal vector in $x \in \Sigma_k, l = l(x)$ a tangential vector of l_k with any prescribed orientation, see [7, 9, 10, 24]. In [17], Theorem 6.2, Remark 3.5 we have proved that the map K from (1) - (3) can be extended to a bounded linear operator

$$\tilde{K} : C^0(\overline{\Omega}) \to V, \tag{9}$$

where the subspace

$$V = \{u \in C^0(\overline{\Omega}) | [u]_\ell < \infty, \, u \cdot N|_S = 0\}$$

of $C^0(\overline{\Omega})$ is equipped with the norm $|u|_\ell = |u|_0 + [u]_\ell$, $[u]_\ell$ denoting the quasi-Lipschitz bound from (5). Here the compatibility conditions (4) and $\operatorname{div} w = 0$ are unnecessary since we do not require $w = \operatorname{rot} \tilde{K} w$.

In [17] the boundedness property of $\tilde{K}$ has given us the decisive tool for proving

Theorem 2.1. *Let the prescribed initial value w_0 be one times continuously differentiable, the vector function $Z(x) = x + \varepsilon w_0(x)$ for $x \in \overline{\Omega}$ taking its values in $\overline{\Omega}$, where $\varepsilon \neq 0$ denotes a constant. Then the initial value problem*

$$\begin{aligned} \frac{\partial}{\partial t} w + v \cdot \nabla w &= \frac{1}{\varepsilon} \Big\{ v(t, x + \varepsilon w(t,x)) \\ &\quad - v(t,x) \Big\}, \qquad (t,x) \in J \times \overline{\Omega}, \qquad (10) \\ w(0,x) &= w_0(x) \qquad\qquad\qquad\qquad\qquad (11) \end{aligned}$$

with $v = \tilde{K} w$ (the function v being somehow continuously extended to $J \times \mathbb{R}^3$) has a unique global solution $w \in C^1(J \times \overline{\Omega})$. The function w can be approximated by iteration of a contracting map T and depends continuously on w_0.

The construction of the map T is similar as in [15], where the Cauchy problem of (10) in $\mathbb{R}^3$ without the boundary condition for the velocity field v has been solved, the initial value w_0 having compact support. The fulfilling of (2) requires the new potential theoretic tools developed in [17].

3 Some ideas of the proof (c.p. [18])

For any continuous $v \in C^0(J \times \overline{\Omega})$, the boundary condition (2) $v \cdot N = 0$ on S ensures global existence for all $t \in J$ of solutions $x(t)$ to the initial value problem

$$\frac{d}{dt} = v(t,x), \quad x(s) = x_s \in \overline{\Omega} \tag{12}$$

starting at $x_s \in \overline{\Omega}$ at time $s \in J$, [14]. The quasi–Lipschitz condition being a special uniqueness condition [13, 25], for any $v \in C^0(J \times \overline{\Omega})$ which fulfills (2) and $[v(t,\cdot)]_\ell \leq c$ uniformly, the global flow

$$x = X(t,s,x_s) = (Lv)(t,s,x_s) \tag{13}$$

of (12) is uniquely defined for $(t, s, x_s) \in J \times J \times \overline{\Omega}$, X describing the unique particles' pathes $x(t) = X(t, s, x_s)$ from (12). For all $t, s \in J$, $X(t, s, \cdot)$ represents a topological map of $\overline{\Omega}$ onto itself, X fulfilling $X^{-1}(t, s, \cdot) = X(s, t, \cdot), X(s, s, x) = x$, [2].

Firstly let $v \in C^0(J \times \overline{\Omega})$ be given, v being continuously differentiable with respect to $x \in \overline{\Omega}$ and fulfilling (2). Introducing the Lagrangean representation $\hat{w}(t, x_0) = w(t, X(t, 0, x_0))$, from the identity

$$\left(\frac{\partial}{\partial t}\hat{w}(t, \cdot)\right) \circ X^{-1}(t, 0, x) = \frac{\partial}{\partial t}w + v \cdot \nabla w \quad (14)$$

for the material derivative we easily see that the partially discretized Helmholtz equation (10) is equivalent to Cauchy's discretized vorticity equation

$$\begin{aligned} w(t, x) &= \frac{1}{\varepsilon}\{X(t, 0, Z(\cdot)) - X(t, 0, \cdot)\} \\ &\quad \circ X(0, t, x) = (HX)(t, x), \end{aligned} \quad (15)$$

where $Z(x) = x + \varepsilon w_0(x) \in \overline{\Omega}$, and that $x + \varepsilon w(t, x) \in \overline{\Omega}$ holds for all $x \in \overline{\Omega}, t \in J$. Recalling the representations $X = Lv, v = \tilde{K}w$, from (15) we are led to the fixed–point equation

$$w = HL\tilde{K}w \equiv Tw. \quad (16)$$

The existence of a unique solution w depending continuously on w_0 follows by the contracting mapping principle which we apply on T in the space

$$\begin{aligned} C_{w_0} = \Big\{ & f \in C^0(J \times \overline{\Omega});\ |f|_0 \le \frac{\text{diam}\ \Omega}{|\varepsilon|}, \\ & [f(t, \cdot)]_\alpha \le M_1, f(0, \cdot) = w_0 \Big\} \end{aligned} \quad (17)$$

with suitable values M_1, α depending on $|\varepsilon|$, $|w_0|_0$, $|\nabla w_0|_0$ and the length of J.

Then from the regularization properties of the maps $\tilde{K}$ and L we find that the solution $w = (Tw) \in C_{w_0}$ is one times continuously differentiable with respect to t and x, see [18].

References:

[1] E. B. Bykhovskii, N. V. Smirnov, On orthogonal decomposition of the space of vector-valued functions quadratically summable in a given domain, *Proc. Math. Inst. Steklov* 59, 1960, pp. 6–36.

[2] E. A. Coddington, N. Levinson, *Theory of Ordinary Differential Equations,* McGraw-Hill, New York, 1955.

[3] J. Deng, T. Y. Hou, X. Yu, Geometric properties and nonblowup of $3D$ incompressible Euler flow, *Comm. Partial Diff. Eq.* 30, 2005, 225–243.

[4] N. M. Günter, *Theory of potentials and its application to basic problems of mathematical physics,* Moscow, 1953.

[5] E. Hölder, Über die unbeschränkte Fortsetzbarkeit einer stetigen ebenen Bewegung in einer unbegrenzten Flüssigkeit, *Math. Z.* 37, 1933, 727–738.

[6] T. Kato, On classical solutions of the two-dimensional nonstationary Euler equations, *Arch. Rat. Mech. Anal.* 25, 1967, 188–200.

[7] R. Kress, Grundzüge einer Theorie der harmonischen Vektorfelder, in: B. Brosowski, E. Martensen eds., *Methoden und Verfahren der Mathematischen Physik* 2, 1969, 49–83.

[8] O. A. Ladyzenskaya, V. A. Solonnikov, Resolution of some evolution problem of magnetohydrodynamics for a viscous incompressible fluid, *Proc. Math. Inst. Steklov* 59, 1960, 282–285.

[9] E. Martensen, *Potentialtheorie,* Teubner, Stuttgart 1968.

[10] E. Martensen, Vektorielle Differentialgleichungen der Potentialtheorie, *ZAMM* 52, 1972, T17–T24.

[11] T. Ogawa, Y. Taniuchi, A note on blow-up criterion to the 3-D Euler equations in a bounded domain, *J. Math. Fluid Mech.* 5, 2003, 17–23.

[12] T. Ogawa, Y. Taniuchi, On blow-up criteria of smooth solutions to the 3-D Euler equations in a bounded domain, *J. Differential Equations* 190, 2003, 39–63.

[13] W. F. Osgood, Beweis der Existenz einer Lösung der Differentialgleichung $dy/dx = f(x, y)$ ohne Hinzunahme der Cauchy-Lipschitzschen Bedingung, *Monatshefte Math. Phys.* 9, 1898, 331–345.

[14] R. Rautmann, A criterion for global existence in case of ordinary differential equations, *Applicable Analysis* 2, 1972, 187–194.

[15] R. Rautmann, Das Cauchyproblem der Helmholtzschen Wirbelgleichung mit einer Differenzennäherung, in: *Recent developments in theoretical and experimental fluid mechanics,* U. Müller, K. G. Roesner, B. Schmidt eds., Springer 1979, 295–308.

[16] R. Rautmann, Ein Näherungsverfahren für spezielle parabolische Anfangswertaufgaben mit Operatoren, in: *Numerische Lösung nichtlinearer partieller Differential- und Integralgleichungen,* R. Ansorge, W. Törnig eds., Springer, Lecture Notes in Math. 267, 1972, 187–231.

[17] R. Rautmann, V. A. Solonnikov, Quasi-Lipschitz condition in potential theory, *Math. Nachr.* 278, 2005, 485–505.

[18] R. Rautmann, V. Solonnikov, Helmholtz's vorticity transport equation with partial discretization in bounded 3-dimensional domains, to appear.

[19] R. M. Redheffer, The theorems of Bony and Brezis on flow-invariant sets, *Amer. Math. Monthly* 79, 1972, 740–747.

[20] J. Schauder, Potentialtheoretische Untersuchungen, *Math. Z.* 33, 1931, 602–640.

[21] J. Serrin, Mathematical principles of classical fluid mechanics, in: Flügge (ed), *Handbuch der Physik* 8, Springer Berlin, 1959, 125–263.

[22] V. A. Solonnikov, Overdetermined elliptic boundary value problems, *Zap. Nauch. Sem. LOMI* 21, 1971, 112–158.

[23] V. A. Solonnikov, On estimates of Green's tensors for some boundary value problems, *Doklady Acad. Sci. USSR* 130, 1960, 988–991.

[24] W. von Wahl, *Vorlesung über das Außen raumproblem für die instationären Gleichungen von Navier-Stokes,* Veröffentlichungen des Sonderforschungsbereichs 256, Institut für Angewandte Mathematik der Universität Bonn, Vol. 11, 1989.

[25] W. Walter, *Differential and Integral Inequalities*, Springer, Berlin 1970.

[26] J. Weissinger, Zur Theorie und Anwendung des Iterationsverfahrens, *Math. Nachr.* 8, 1952, 193–212.

[27] W. Wolibner, Un théoréme sur l'existence du mouvement plan d'un fluide parfait, homogéne, incompressible, pendant un temps infinement longue, *Math. Z.*, 37, 1933, 698–726.

[28] W. I. Yudovič, Nonstationary flows of an ideal incompressible fluid, *Zh. Vyc. Mat.* 3, 1963, 1032–1066.