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On a Suitable Weak Solution of the Navier–Stokes Equation with the Generalized Impermeability Boundary Conditions

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Abstract: - We prove that the Navier–Stokes initial–boundary value problem with the generalized impermeability boundary conditions has a global in time suitable weak solution (in the sense of [2]) that satisfies the generalized energy inequality up to the boundary of the flow field. We suggest the method how the solution can be constructed.

Key-Words: - Navier–Stokes equation, Suitable weak solution, Generalized energy inequality

1 Introduction

The Navier–Stokes system

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \nu \Delta \boldsymbol{u} + \boldsymbol{f}, \tag{1}$$

$$\operatorname{div} \boldsymbol{u} = 0, \tag{2}$$

is mostly studied with the no–slip condition

$$\boldsymbol{u} = \boldsymbol{0} \tag{3}$$

on the fixed boundary of the flow field. The Navier–Stokes equation (1) expresses the conservation of momentum and the equation of continuity (2) expresses the conservation of mass in a viscous incompressible fluid. $\boldsymbol{u} = (u_1, u_2, u_3)$ is the velocity, p denotes the pressure, ν is the kinematic coefficient of viscosity and $\boldsymbol{f}$ is the external body force. We shall further assume, for simplicity, that $\nu = 1$. Although many important questions still remain open, it is possible to say that the theory of the system (1), (2) with boundary condition (3) is deeply elaborated. The survey of main results on non–steady solutions can be found e.g. in paper [3] by G. P. Galdi. The Navier–Stokes equation was originally derived in the 19th century by physicists under the a priori assumption on smoothness of its solutions. However, in spite of an enormous effort of many scientists, the question of the global in time existence of a smooth solution for arbitrarily large smooth initial data has not been solved yet and it belongs to most challenging open problems of today's theory of partial differential equations. It is only known that the problem (1)–(3) with an appropriate initial condition

$$\boldsymbol{u}\,|_{t=0} = \boldsymbol{u}_0 \tag{4}$$

has a global in time weak solution. (See e.g. [3] for the exact definition.) A series of papers shows that the weak solution can be constructed so that the set of its singular points, if it is not empty, is in some sense relatively small. The most exhausting result was given by L. Caffarelli, R. Kohn and L. Nirenberg in [2]. The authors introduced the notion of a "suitable weak solution" of (1)–(4) on $Q_T \equiv \Omega \times (0,T)$ (where Ω is a domain in $\mathbb{R}^3$ and $T > 0$ is given) as a pair of measurable functions $\boldsymbol{u}$, p which fulfill equations (1) and (2) in the sense of distributions, $\boldsymbol{u}$ is a weak solution of (1)–(4), $p \in L^{5/4}(Q_T)$ and $\boldsymbol{u}$ and p satisfy a so called generalized energy inequality in the interior of Q_T. The inequality will be shown in Section 2. In [2], L. Caffarelli, R. Kohn and L. Nirenberg proved the global in time existence of a suitable weak solution under certain restriction on the smoothness of the initial data, but not on the size of the data. Nevertheless, the restriction was later removed by Y. Taniuchi (see [10]) and thus, the suitable weak solution becomes a natural type of a weak solution of the problem (1)–(4). The generalized energy inequality plays a fundamental role in treatments of local properties of the suitable weak solution. It enables to derive the important information on the 1–dimensional Hausdorff measure of the set $S(\boldsymbol{u})$ of eventual singular points of the suitable weak solution $(\boldsymbol{u};\, p)$ in Q_T: the measure equals zero.

H. Bellout, J. Neustupa and P. Penel [1] have shown that a systematic theory of the Navier–Stokes equation can also be created with the boundary conditions

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0, \;\; \mathbf{curl}\, \boldsymbol{u} \cdot \boldsymbol{n} = 0, \;\; \mathbf{curl}^2 \boldsymbol{u} \cdot \boldsymbol{n} = 0 \tag{5}$$

on $\partial\Omega$, which were later called the *generalized imper-*

meability boundary conditions. ($\boldsymbol{n}$ is the outer normal vector on $\partial\Omega$.) Most of qualitative results from the theory of the Navier–Stokes equation with the Dirichlet boundary condition (3) are also in some form true for the Navier–Stokes equation with boundary conditions (5) and moreover, conditions (5) enable to derive some finer results on the regularity of a solution – see the papers [1], [7] and [8]. (The last paper also discusses the physical sense of the boundary conditions (5) and the relation between these boundary condition and the homogeneous Dirichlet boundary condition (3) and it was submitted to the same journal as the presented paper.)

The aim of this paper is to show that a suitable weak solution of the problem (1), (2), (4) with the generalized impermeability boundary conditions (5) can be constructed so that it satisfies the generalized energy inequality not only in the interior of Q_T, but also up to the boundary.

2 Notation, Definitions and Formulation of the Main Result

We assume that Ω is a bounded domain in $\mathbb{R}^3$ with the boundary $\partial\Omega$ of the class $C^{2+\mu}$ for some $\mu > 0$. We shall use the notation:

- H is a closure of $\{\boldsymbol{v} \in C_0^\infty(\Omega)^3;\ \operatorname{div}\boldsymbol{v} = 0\}$ in $L^2(\Omega)^3$. It represents the space of divergence–free (in the sense of distributions) vector functions in $L^2(\Omega)^3$ whose normal component on $\partial\Omega$ equals zero in the sense of traces.
- P_H is the orthogonal projection of $L^2(\Omega)^3$ onto H.
- $H^\perp$ is the orthogonal complement to H in $L^2(\Omega)^3$. It coincides with $\{\nabla\varphi;\ \varphi \in W^{1,2}(\Omega)\}$.
- $(.\,,\,.)_{0,2}$ is the scalar product in $L^2(\Omega)^3$ and in H.
- $\|\,.\,\|_{0,s}$ denotes the norm in $L^s(\Omega)$ and $\|\,.\,\|_{k,s}$ is the norm in the Sobolev space $W^{k,s}(\Omega)$.
- $|||\,.\,|||_{r;\,0,s}$ is the norm in the anisotropic Lebesgue space $L^r(0,T;\ L^s(\Omega))$ and $|||\,.\,|||_{r;\,k,s}$ is the norm in the space $L^r(0,T;\ W^{k,s}(\Omega))$.
- The norms of vector functions will be denoted in the same way as the norms of scalar functions.
- D^1 is the set of functions $\boldsymbol{u} \in W^{1,2}(\Omega)^3 \cap H$ such that $(\mathbf{curl}\,\boldsymbol{u}\cdot\boldsymbol{n})|_{\partial\Omega} = 0$ in the sense of traces. It is a closed subspace of $W^{1,2}(\Omega)^3$.
- D^{-1} is the dual to D^1. The duality between the elements of D^{-1} and D^1 is denoted by $\langle .\,,\,.\rangle$. The norm in D^{-1} is denoted by $\|\,.\,\|_{-1,2}$.
- $A = \mathbf{curl}\,|_{D^1}$ (Thus, $D^1 = D(A)$.)
- $D^2 \equiv D(A^2)$. It is proved in [1] that $D^2 = \{\boldsymbol{v} \in W^{2,2}(\Omega)^3;\ \operatorname{div}\boldsymbol{v} = 0$ a.e. in Ω and $\mathbf{curl}^j\boldsymbol{u}\cdot\boldsymbol{n})|_{\partial\Omega} = 0$ for $j = 0, 1, 2$ in the sense of traces$\}$.
- Note that $A^2 = \mathbf{curl}^2 = -\Delta$ on D^2.

The next lemma reviews some results from [1]. The self–adjointness of operator A was also earlier proved by Z. Yosida, Y. Giga [12] and R. Picard [9] and some other of its properties directly follow from a series of articles of O. A. Ladyzhenskaya, V. A. Solonnikov and their co–workers on operator $\mathbf{curl}$; let us cite e.g. [5].

Lemma 1 *a)* $D^1 = P_H\,W_0^{1,2}(\Omega)^3 = \{\boldsymbol{v} = \boldsymbol{v}_0 + \nabla\varphi;\ \boldsymbol{v}_0 \in W_0^{1,2}(\Omega)^3,\ \Delta\varphi = -\operatorname{div}\boldsymbol{v}_0$ in Ω and $\partial\varphi/\partial\boldsymbol{n}\,|_{\partial\Omega} = 0\}$
b) A is a selfadjoint operator in H with a compact resolvent.
c) The norm $\|\,.\,\|_{k,2}$ is equivalent with the norm $\|A^k\,.\,\|_{0,2}$ in D^k for $k = 1,\ 2$.

Lemma 1 implies that A^2 is a positive selfadjoint operator with a compact resolvent in H. Its eigenvalues can be ordered into a non–decreasing sequence λ_i ($i = 1, 2, \dots$) and the corresponding eigenfunctions $\boldsymbol{e}_i$ form an orthonormal complete system in H which is also orthogonal in D^1 and in D^2. The spaces D^k ($k = 1,\ 2$) can be characterized by the identities

$$D^k = \Big\{\boldsymbol{v} = \sum_{i=1}^{+\infty}\alpha_i\,\boldsymbol{e}_i;\ \sum_{i=1}^{+\infty}\alpha_i^2\lambda_i^k < +\infty\Big\}.$$

The spaces $L^2(0,T;\ D^k)$ ($k = 1,\ 2$) coincide with the sets

$$\Big\{\boldsymbol{w} = \sum_{i=1}^{+\infty}a_i(t)\,\boldsymbol{e}_i;\ \sum_{i=1}^{+\infty}\lambda_i^k\Big(\int_0^T a_i^2\Big) < +\infty\Big\}.$$

Definition 2 Let $\boldsymbol{f} \in L^q(Q_T)^3$ for some $q > 5/2$, $\operatorname{div}\boldsymbol{f} = 0$ in Q_T in the sense of distributions and $\boldsymbol{u}_0 \in H$. The pair $(\boldsymbol{u};\,p)$ is called a *suitable weak solution* of (1), (2), (4), (5) if $\boldsymbol{u} \in L^2(0,T;\ D^1) \cap L^\infty(0,T;\ H)$, $\boldsymbol{u}(.\,,t) \longrightarrow \boldsymbol{u}_0$ weakly in H as $t \to 0_+$, $p \in L^{5/4}(Q_T)$, the Navier–Stokes equation (1) is satisfied in the sense of distributions in Q_T and

$$\int_{\Omega\times\{t_2\}}|\boldsymbol{u}|^2\phi + 2\int_{t_1}^{t_2}\int_\Omega|A\boldsymbol{u}|^2\phi \le \int_{\Omega\times\{t_1\}}|\boldsymbol{u}|^2\phi$$
$$+\int_{t_1}^{t_2}\int_\Omega\Big(|\boldsymbol{u}|^2(\partial_t\phi + \Delta\phi) + (|\boldsymbol{u}|^2 + 2p)\boldsymbol{u}\cdot\nabla\phi$$
$$-\ 2u_iu_j\,\partial_i\partial_j\phi + 2\boldsymbol{f}\cdot\boldsymbol{u}\,\phi\Big) - \int_{t_1}^{t_2}\int_{\partial\Omega}|\boldsymbol{u}|^2\,\frac{\partial\phi}{\partial\boldsymbol{n}} \qquad (6)$$

for every $\phi \in C^\infty(\overline{Q_T})$ such that $\phi \ge 0$, ϕ is only a function of t on $\partial\Omega \times [0,T]$ and for a.a. $t_1 \in [0,T)$ (including $t_1 = 0$) and all $t_2 \in (t_1, T]$.

Inequality (6) is the *generalized energy inequality* mentioned in Section 1. It can be formally obtained by multiplying equation (1) by $\boldsymbol{u}\,\phi$ and integrating by parts over $\Omega \times (t_1, t_2)$.

Definition 2 is analogous to the definition of the suitable weak solution given by L. Caffarelli, R. Kohn and L. Nirenberg in [2], but it is not quite identical. The first reason is that the boundary condition (3) is considered in [2], while we use the boundary conditions (5). The second reason is that function ϕ is required to have a compact support in $\Omega \times [0,T)$ in [2], while we admit a wider class of test functions ϕ in Definition 2. Thus, our suitable weak solution extends the notion of the suitable weak solution introduced in [2] because our definition already involves the validity of the generalized energy inequality "up to the boundary". On the other hand, in a special case when ϕ has a compact support in $\Omega \times [0,T)$, using the well known identity $|\nabla \boldsymbol{u}|^2 = |\mathbf{curl}\,\boldsymbol{u}|^2 + (\partial_j u_i)(\partial_i u_j)$, we can show that inequality (6) formally coincides with the generalized energy inequality from [2].

The next theorem represents the main result of this paper.

Theorem 3 *Let $\boldsymbol{f} \in L^2(Q_T)^3$ and $\boldsymbol{u}_0 \in D^2$. Then there exists a suitable weak solution $(\boldsymbol{u};\, p)$ of the problem (1), (2), (4), (5), introduced by Definition 2. Moreover, $\boldsymbol{u}$ is a weakly continuous mapping from $[0,T]$ into $L^2(\Omega)^3$ and $p \in L^\alpha(0,T;\, L^\beta(\Omega))$ for arbitrary $\alpha \in (1,2)$, $\beta \in (\frac{3}{2},3)$ such that*

$$\frac{2}{\alpha} + \frac{3}{\beta} = 3.$$

3 Proof of Theorem 3

In order to prove Theorem 3, we shall need several lemmas. The first one is proved in [6].

Lemma 4 *If $g \in L^\infty(0,T;\, L^2(\Omega)) \cap L^2(0,T;\, L^6(\Omega))$ and $2 \le \alpha \le +\infty$, $2 \le \beta \le 6$ and $2/\alpha + 3/\beta \ge \frac{3}{2}$ then*

$$|||g|||_{\alpha;\,0,\beta} \le |||g|||_{2;\,0,2}^{2/\alpha+3/\beta-3/2} \cdot \Big(|||g|||_{\infty;\,0,2} + |||g|||_{2;\,0,6}\Big)^{5/2-(2/\alpha+3/\beta)}. \qquad (7)$$

Particularly, if $2/\alpha + 3/\beta = \frac{3}{2}$ then

$$|||g|||_{\alpha;\,0,\beta} \le |||g|||_{\infty;\,0,2} + |||g|||_{2;\,0,6}\,. \qquad (8)$$

We shall further consider functions on Q_T to be mappings from $(0,T)$ with values in an appropriate function space and we shall therefore prefer e.g. the shorter notation $\boldsymbol{w}(t)$ to $\boldsymbol{w}(\,.\,,t)$. By analogy, we shall write $\boldsymbol{w}'$ instead of $\partial_t \boldsymbol{w}$.

Lemma 5 *Let $\boldsymbol{g} \in L^2(0,T;\, H)$ and $\boldsymbol{w}_0 \in D^1$. Then the non–steady Stokes problem*

$$\boldsymbol{w}' + A^2\boldsymbol{w} = \boldsymbol{g} \qquad (9)$$

$$\boldsymbol{w}(0) = \boldsymbol{w}_0 \qquad (10)$$

has a unique solution $\boldsymbol{w} \in L^2(0,T;\, D^2) \cap L^\infty(0,T;\, D^1)$ such that $\boldsymbol{w}' \in L^2(0,T;\, H)$ and

$$|||\boldsymbol{w}|||_{2;\,2,2} + |||\boldsymbol{w}|||_{\infty;\,1,2} + |||\boldsymbol{w}'|||_{2;\,0,2} \le c_1\, \|\boldsymbol{w}_0\|_{1,2} + c_2\, |||\boldsymbol{g}|||_{2;\,0,2}\,. \qquad (11)$$

Proof: The existence and uniqueness of a solution $\boldsymbol{w} \in L^2(0,T;\, D^2) \cap L^\infty(0,T;\, D^1)$ such that $\boldsymbol{w}' \in L^2(0,T;\, D^{-1})$ follows from [1]. The solution can thus be expressed in the form

$$\boldsymbol{w}(t) = \sum_{i=1}^{+\infty} a_i(t)\, \boldsymbol{e}_i$$

where functions $a_i(t)$ satisfy the initial–value problems

$$a_i' + \lambda_i\, a_i = (\boldsymbol{g}, \boldsymbol{e}_i)_{0,2} \quad \text{a.e. on } (0,T), \qquad (12)$$

$$a_i(0) = (\boldsymbol{w}_0, \boldsymbol{e}_i)_{0,2} \qquad (13)$$

for $i = 1, 2, \dots$ Multiplying equation (12) by $\lambda_i a_i$ and integrating from 0 to t, we obtain

$$\frac{\lambda_i}{2}\, a_i^2(t) + \lambda_i^2 \int_0^t a_i^2 = \frac{\lambda_i}{2}\, a_i^2(0) + \lambda_i \int_0^t (\boldsymbol{g}, \boldsymbol{e}_i)_{0,2}\, a_i$$

$$\le \frac{\lambda_i}{2}\, a_i^2(0) + \frac{\lambda_i^2}{2} \int_0^t a_i^2 + \frac{1}{2} \int_0^t (\boldsymbol{g}, \boldsymbol{e}_i)_{0,2}^2\,,$$

$$\frac{\lambda_i}{2}\, a_i^2(t) + \frac{\lambda_i^2}{2} \int_0^t a_i^2 \le \frac{\lambda_i}{2}\, a_i^2(0) + \frac{1}{2} \int_0^T (\boldsymbol{g}, \boldsymbol{e}_i)_{0,2}^2\,.$$

This implies that

$$\sum_{i=1}^{+\infty} \left[\lambda_i^2 \left(\int_0^T a_i^2 \right) + \sup_{t \in (0,T)} \text{ess}\, \lambda_i\, a_i(t) \right] < +\infty$$

and thus $\boldsymbol{w} \in L^2(0,T;\, D^2) \cap L^\infty(0,T;\, D^1)$. The rest of the proof is obvious. □

Lemma 6 *Let $\boldsymbol{g} \in L^2(0,T;\, H)$, $\boldsymbol{w}_0 \in D^1$ and $\boldsymbol{\psi} \in L^2(0,T;\, D^2)$ such that $\boldsymbol{\psi}' \in L^2(0,T;\, H)$. Then there exists a unique solution $\boldsymbol{w} \in L^2(0,T;\, D^2) \cap L^\infty(0,T;\, D^1)$ of the problem*

$$\boldsymbol{w}' + A^2\boldsymbol{w} + P_H(\boldsymbol{\psi} \cdot \nabla \boldsymbol{w}) = \boldsymbol{g} \qquad (14)$$

$$\boldsymbol{w}(0) = \boldsymbol{w}_0 \qquad (15)$$

on the interval $(0,T)$ such that $\boldsymbol{w}' \in L^2(0,T;\, H)$.

Proof: Let

$$X = \{\boldsymbol{v} \in L^2(0,T;\, D^2);\ \boldsymbol{v}' \in L^2(0,T;\, H)\},$$
$$Y = \{\,[\boldsymbol{g},\boldsymbol{\omega}];\ \boldsymbol{g} \in L^2(0,T;\, H),\ \boldsymbol{\omega} \in D^1\}$$

be the Banach spaces with the norms

$$|||\boldsymbol{v}|||_X = |||\boldsymbol{v}|||_{2;\,2,2} + |||\boldsymbol{v}'|||_{2;\,0,2},$$
$$|||\,[\boldsymbol{g},\boldsymbol{\omega}]\,|||_Y = |||\boldsymbol{g}|||_{2;\,0,2} + \|\boldsymbol{\omega}\|_{1,2}\,.$$

Then due to Lemma 5,

$$S\,:\,\boldsymbol{v} \longrightarrow S\boldsymbol{v} \equiv [\boldsymbol{v}' + A^2\boldsymbol{v},\, \boldsymbol{v}(0)]$$

is a one–to–one operator from X onto Y. The operator $B\boldsymbol{v} = [P_H(\boldsymbol{\psi}\cdot\nabla\boldsymbol{v}),\, \boldsymbol{0}]$ is a compact operator from X into Y and $S+B$ is an injective operator. Hence $S+B$ is a one–to–one operator from X onto Y and consequently, the problem (14), (15) has a unique solution in X. □

Lemma 7 *Let $\boldsymbol{\psi} \in L^2(0,T;\, D^2)$, $\psi' \in L^2(0,T;\, H)$, $\boldsymbol{f} \in L^2(0,T;\, L^2(\Omega)^3)$ and $\boldsymbol{w}_0 \in D^1$. Then there exists a unique solution $(\boldsymbol{w}, q)$ of the problem*

$$\boldsymbol{w}' - \Delta\boldsymbol{w} + \boldsymbol{\psi}\cdot\nabla\boldsymbol{w} + \nabla q = \boldsymbol{f}, \tag{16}$$
$$\boldsymbol{w}(0) = \boldsymbol{w}_0 \tag{17}$$

such that $\boldsymbol{w} \in L^2(0,T;\, D^2) \cap L^\infty(0,T;\, D^1)$, $\boldsymbol{w}' \in L^2(0,T;\, H)$, $\nabla q \in L^2(0,T;\, L^2(\Omega))$ and the average of $q(t)$ on Ω equals zero for a.a. $t \in (0,T)$. Moreover,

$$|||q|||_{\alpha;\,0,\beta} + |||\nabla q|||_{\alpha;\,0,\gamma} \le c_3\Big(|||\boldsymbol{f}|||_{2;\,0,2} + \|\boldsymbol{w}_0\|_{1,2}\Big)\cdot\Big(|||\boldsymbol{\psi}|||_{\infty;\,0,2} + |||\boldsymbol{\psi}|||_{2;\,0,6} + 1\Big) \tag{18}$$

for arbitrary α, β and γ such that $1 < \alpha < 2$, $\frac{3}{2} < \beta < 3$, $1 < \gamma < \frac{3}{2}$ and

$$\frac{2}{\alpha} + \frac{3}{\beta} = 3, \qquad \frac{2}{\alpha} + \frac{3}{\gamma} = 4. \tag{19}$$

Proof: $\boldsymbol{f}$ can be expressed in the form $\boldsymbol{f}^1 + \boldsymbol{f}^2$ where $\boldsymbol{f}^1 \in L^2(0,T;\, H)$ and $\boldsymbol{f}^2 \in L^2(0,T;\, H^\perp)$. Due to Lemma 6, there exists a unique solution $\boldsymbol{w}$ of the problem (14), (15) with $\boldsymbol{g} = \boldsymbol{f}^1$. Moreover, applying the standard estimate to equation (14), we obtain:

$$|||\boldsymbol{w}|||_{2;\,1,2} + |||\boldsymbol{w}|||_{\infty;\,0,2} \le c_4\,\|\boldsymbol{w}_0\|_{0,2} + c_5\,|||\boldsymbol{f}^1|||_{2;\,0,2}\,. \tag{20}$$

This estimate, together with the information on $\boldsymbol{\psi}$, implies that $\boldsymbol{\psi}\cdot\nabla\boldsymbol{w} \in L^2(0,T;\, L^2(\Omega)^3)$. Put $\nabla q = \boldsymbol{f}^2 - (I-P_H)(\boldsymbol{\psi}\cdot\nabla\boldsymbol{w})$. Then $\nabla q \in L^2(0,T;\, L^2(\Omega)^3)$ and the pair $\boldsymbol{w}$, q solves (16), (17). Choosing q so that its average on Ω is zero at a.a. times $t \in (0,T)$, we achieve that $q \in L^2(0,T;\, L^6(\Omega))$. Using the inequality

$$|||\boldsymbol{\psi}\cdot\nabla\boldsymbol{w}|||_{\alpha;\,0,\gamma} \le c_6\,|||\nabla\boldsymbol{w}|||_{2;\,0,2}\,|||\boldsymbol{\psi}|||_{r;\,0,s} \le c_7\,|||\nabla\boldsymbol{w}|||_{2;\,0,2}\Big(|||\boldsymbol{\psi}|||_{\infty;\,0,2} + |||\boldsymbol{\psi}|||_{2;\,0,6}\Big)$$

where

$$r = \frac{2\alpha}{2-\alpha}, \qquad s = \frac{2\gamma}{2-\gamma}$$

and α, β, γ satisfy (19), we can also derive the estimate

$$|||(I-P_H)(\boldsymbol{\psi}\cdot\nabla\boldsymbol{w})|||_{\alpha;\,0,\gamma} \le c_8\,|||\nabla\boldsymbol{w}|||_{2;\,0,2}\Big(|||\boldsymbol{\psi}|||_{\infty;\,0,2} + |||\boldsymbol{\psi}|||_{2;\,0,6}\Big).$$

This and (20) enable to obtain the estimate of the norm of ∇q in (18). The estimate of q can now be easily derived using mainly the fact that the integral of q on Ω equals zero. □

Multiplying equation (16) by $2\phi\boldsymbol{w}$, where ϕ is a non–negative function in $C^\infty(\overline{Q_T})$ such that it depends only on t on $\partial\Omega\times[t_1,t_2]$, and integrating on $\Omega\times[t_1,t_2]$, where $t_1 \in [0,T)$, $t_2 \in (t_1,T]$, we can arrive at the generalized energy equality

$$\int_{\Omega\times\{t_2\}}|\boldsymbol{w}|^2\phi + 2\int_{t_1}^{t_2}\!\!\int_\Omega |A\boldsymbol{w}|^2\phi = \int_{\Omega\times\{t_1\}}|\boldsymbol{w}|^2\phi + \int_{t_1}^{t_2}\!\!\int_\Omega\Big(|\boldsymbol{w}|^2(\partial_t\phi + \Delta\phi) + (|\boldsymbol{w}|^2\boldsymbol{\psi} + 2q\boldsymbol{w})\cdot\nabla\phi - 2w_iw_j\,\partial_i\partial_j\phi + 2\boldsymbol{f}\cdot\boldsymbol{w}\,\phi\Big) - \int_{t_1}^{t_2}\!\!\int_{\partial\Omega}|\boldsymbol{w}|^2\,\frac{\partial\phi}{\partial\boldsymbol{n}} \tag{21}$$

Let $\boldsymbol{\varphi} \in C(0,T;\, H)$ and $n \in \mathbb{N}$. Put $\delta_n = T/n$ and

$$\Psi_n(\boldsymbol{\varphi})(t) = \begin{cases} \boldsymbol{\varphi}(0) & \text{for } t \in (0,\delta_n), \\ \boldsymbol{\varphi}(t-\delta_n) & \text{for } t \in [\delta_n, T). \end{cases}$$

Obviously, $\Psi_n(\boldsymbol{\varphi}) \in C(0,T;\, H)$.

Let $(\boldsymbol{w}_n;\, q_n)$ be the solution of the problem

$$\boldsymbol{w}_n' + A^2\boldsymbol{w}_n + \Psi_n(\boldsymbol{w}_n)\cdot\nabla\boldsymbol{w}_n + \nabla q = \boldsymbol{f} \tag{22}$$
$$\boldsymbol{w}_n(0) = \boldsymbol{u}_0 \tag{23}$$

on $(0,T)$. We can easily verify that $\boldsymbol{w}_n$, $\Psi_n(\boldsymbol{w}_n) \in L^2(0,\delta_n;\, D^2)$. Applying successively Lemma 7 on the time intervals $(k\delta_n, (k+1)\delta_n)$, $k = 1,\dots,n-1$, we can even obtain that $\boldsymbol{w}_n$, $\Psi_n(\boldsymbol{w}_n) \in L^2(0,T;\, D^2)$. Standard energy estimates, applied to solution $\boldsymbol{w}_n$, show that all the norms $|||\boldsymbol{w}_n|||_{2;\,1,2}$, $|||\boldsymbol{w}_n|||_{\infty;\,0,2}$ and consequently also $|||\Psi_n(\boldsymbol{w}_n)|||_{2;\,1,2}$, $|||\Psi_n(\boldsymbol{w}_n)|||_{\infty;\,0,2}$ can be estimated by a constant independent of n. Similarly, using estimate (18), we can derive that the norms $\|\|q_n\|\|_{\alpha;\,0,\beta}$ and $\|\|\nabla q_n\|\|_{\alpha;\,0,\gamma}$ are estimated by a constant independent of n. (α, β and γ satisfy (19).) Moreover, we can directly deduce from equation (22) that the norm $\|\|\boldsymbol{w}_n'\|\|_{4/3;\,-1,2}$ can be estimated by c_9 which is independent of n. ($\|\|\,.\,\|\|_{4/3;\,-1,2}$ is the norm in the space $L^{4/3}(0,T;\,[W^{1,2}(\Omega)^3]^*)$ where $[W^{1,2}(\Omega)^3]^*$ denotes the dual to $W^{1,2}(\Omega)^3$.) Using finally this estimate and the structure of $\Psi_n(\boldsymbol{w}_n)$, we can derive that the same also holds about the norm of $\Psi_n(\boldsymbol{w}_n)'$ in $L^{4/3}(0,T;\,[W^{1,2}(\Omega)^3]^*)$. The space $L^2(0,T;\,W^{1,2}(\Omega)^3)$ is reflexive, hence there exists a sub–sequence of $\{\boldsymbol{w}_n\}$ (we denote the sub–sequence in the same way in order to preserve a simple notation) and $\boldsymbol{u}, \boldsymbol{u}^* \in L^2(0,T;\,W^{1,2}(\Omega)^3)$ such that

$$\boldsymbol{w}_n \;\to\; \boldsymbol{u}, \tag{24}$$

$$\Psi_n(\boldsymbol{w}_n) \;\to\; \boldsymbol{u}^*; \tag{25}$$

both (24) and (25) represent the weak convergence in $L^2(0,T;\,W^{1,2}(\Omega)^3)$. The information on a strong convergence in some spaces can be deduced from the boundedness of the time derivatives of $\boldsymbol{w}_n$ and $\Psi_n(\boldsymbol{w}_n)$. Indeed, applying Lions' lemma (see e.g. R. Temam [11], Theorem 2.1, Chap. III), we obtain that

$$\boldsymbol{w}_n \to \boldsymbol{u} \quad \text{strongly in } L^2(0,T;\,H), \tag{26}$$

$$\boldsymbol{w}_n \to \boldsymbol{u} \quad \text{strongly} \in L^2(0,T;\,L^3(\partial\Omega)^3), \tag{27}$$

$$\Psi_n(\boldsymbol{w}_n) \to \boldsymbol{u}^* \quad \text{strongly} \in L^2(0,T;\,H). \tag{28}$$

Lemma 8 *$\boldsymbol{u}$ and $\boldsymbol{u}^*$ from (24)–(28) satisfy*

$$\boldsymbol{u} \,=\, \boldsymbol{u}^*. \tag{29}$$

Proof: $\{\boldsymbol{w}_n\}$ and $\{\Psi_n(\boldsymbol{w}_n)\}$ are relatively compact sets in $L^2(0,T;\,H)$. Using [4], Theorem 2.13.1, condition(ii), we get that the components w_{n1}, w_{n2}, w_{n3} of $\boldsymbol{w}_n$ are 2–mean equicontinuous functions, i.e. to each $\epsilon > 0$ there exists $\delta > 0$ such that for all $h \in \mathbb{R}$, $|h| < \delta$,

$$\int_{Q_T} |w_{nk}(\boldsymbol{x}, t+h) - w_{nk}(\boldsymbol{x}, t)|^2 \,<\, \epsilon^2 \tag{30}$$

for $k = 1, 2, 3$. (If necessary, w_{nk} are defined to be equal to zero outside Q_T). The inequalities in (30) and the way how functions Ψ_n are defined imply (29). □

The solution $(\boldsymbol{w}_n;\,q_n)$ of (22), (23) naturally satisfies the same generalized energy inequality as (21).

Using Lemma 4, (24) and (26), we can derive that

$$\boldsymbol{w}_n \to \boldsymbol{u} \quad \text{strongly in } L^r(0,T;\,L^s(\Omega)^3)$$

for all $r \in (2,\infty)$, $s \in (2,6)$ such that

$$\frac{2}{r} + \frac{3}{s} \,>\, \frac{3}{2}.$$

Thus, using the boundedness of the sequence $\{q_n\}$ (respectively $\{\nabla q_n\}$) in the space $L^\alpha(0,T;\,L^\beta(\Omega))$ (respectively in $L^\alpha(0,T;\,L^\gamma(\Omega)^3)$) (see (19)), we can deduce that there exists $p \in L^\alpha(0,T;\,L^\beta(\Omega))$, such that $\nabla p \in L^\alpha(0,T;\,L^\gamma(\Omega)^3)$ (where α, β and γ satisfy (19)) so that

$$q_n \to p \quad \text{weakly in } L^\alpha(0,T;\,L^\beta(\Omega)), \tag{31}$$

$$\nabla q_n \to \nabla p \quad \text{weakly in } L^\alpha(0,T;\,L^\gamma(\Omega)^3). \tag{32}$$

Proof of Theorem 3: Applying the standard procedure which is explained e.g. in [11], Chapter III, in the proof of Theorem 3.1, we can show that function $\boldsymbol{u}$ is a weak solution of the problem (1)–(4). Furthermore, function p from (31) and (32) is an associated pressure. (26) implies that

$$\|\boldsymbol{w}_n(t)\|_H \to \|\boldsymbol{u}(t)\|_H \tag{33}$$

for a.a. $t \in (0,T)$ for a sub–sequence of $\{\boldsymbol{w}_n\}$. (We preserve the same notation for the sub–sequence.) (24) implies that

$$\lim_{n\to+\infty} \inf \int_{t_1}^{t_2} \int_\Omega |A\boldsymbol{w}_n|^2 \;\geq\; \int_{t_1}^{t_2} \int_\Omega |A\boldsymbol{u}|^2$$

for all t_1, $t_2 \in [0,T]$ such that $t_1 < t_2$. It is not difficult to get a slightly modified statement

$$\lim_{n\to+\infty} \inf \int_{t_1}^{t_2} \int_\Omega |A\boldsymbol{w}_n|^2\,\phi \;\geq\; \int_{t_1}^{t_2} \int_\Omega |A\boldsymbol{u}|^2\,\phi$$

(for all non–negative functions $\phi \in C^\infty(\overline{Q_T})$). This inequality and all types of convergence (24), (26), (27), (28), (31), (32), (33) enable the limit transition for $n \to +\infty$ in the generalized energy inequality (21), written down for the solution $(\boldsymbol{w}_n;\,q_n)$ of (22), (23). Thus, we obtain inequality (6) for almost all t_1, $t_2 \in (0,T)$ such that $t_1 < t_2$. Using the semi–lower continuity of the function

$$t \in [0,T] \;\longrightarrow\; \int_{\Omega\times\{t\}} |\boldsymbol{u}|^2\,\phi,$$

we can extend the validity of (6) to almost all $t_1 \in [0,T)$ and all $t_2 \in (0,T]$ such that $t_1 < t_2$. Moreover, as (21) holds with $t_1 = 0$, we can deduce that (6) also holds for $t_1 = 0$. This completes the proof of Theorem 3. □

Remark 9 Notice that the so called strong energy inequality

$$\int_{\Omega\times\{t_2\}} |\boldsymbol{u}|^2 + 2\int_{t_1}^{t_2}\int_\Omega |A\boldsymbol{u}|^2 \leq \int_{\Omega\times\{t_1\}} |\boldsymbol{u}|^2 + 2\int_{t_1}^{t_2}\int_\Omega \boldsymbol{f}\cdot\boldsymbol{u} \tag{34}$$

follows from (6) by the choice $\phi \equiv 1$.

4 Conclusion

As we have already mentioned, L. Caffarelli, R. Kohn and L. Nirenberg [2] used the "interior" version of the generalized energy inequality in the study of the 1–dimensional Hausdorff measure of the set of the interior singular points of the suitable weak solution. The presented paper shows that the generalized impermeability boundary conditions (5) enable the extension of the generalized energy inequality "up to the boundary". By analogy with L. Caffarelli, R. Kohn and L. Nirenberg, the extended version of the generalized energy inequality can be further applied to the treatment of the 1–dimensional Hausdorff measure of all singular points (interior as well as boundary) of the suitable weak solution which will be described in a forthcoming paper. This is the main reason why the "up to the boundary" version of the generalized energy inequality (6) is important.

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References:

[1] H. Bellout, J. Neustupa and P. Penel, On the Navier-Stokes Equation with Boundary Conditions Based on Vorticity, *Math. Nachr.,* Vol. 269–270, 2004, pp. 59–72.

[2] L. Caffarelli, R. Kohn and L. Nirenberg, Partial Regularity of Suitable Weak Solutions of the Navier–Stokes Equations, *Comm. on Pure and Applied Math.,* Vol. 35, 1982, pp. 771–831.

[3] G. P. Galdi, An Introduction to the Navier–Stokes Initial–Boundary Value Problem, in *Fundamental Directions in Math. Fluid Mechanics,* ed. G. P. Galdi, J. Heywood, R. Rannacher, Birkhauser, Basel 2000, pp. 1–98.

[4] A. Kufner, O. John and S. Fučík, *Function Spaces,* Academia, Prague 1979.

[5] O. A. Ladyzhenskaya and V. A. Solonnikov, Solution of Some Non–Stationary Problems of Magneto–Hydrodynamics for Viscous Incompressible Fluids, *Trudy Math. Inst. V. A. Steklova,* special volume on "Mathematical Problems in Hydrodynamics and Magnetohydrodynamics", Vol. 59, 1960, pp. 115–173.

[6] J. Neustupa and P. Penel, Anisotropic and Geometric Criteria for Interior Regularity of Weak Solutions to the 3D Navier–Stokes Equations, in *Mathematical Fluid Mechanics, Recent Results and Open Problems,* ed. J. Neustupa, P. Penel, Birkhauser, Basel 2001, pp. 237–268.

[7] J. Neustupa and P. Penel, On Regularity of a Weak Solution to the Navier–Stokes Equation with Generalized Impermeability Boundary Conditions, Preprint Université de Sud–Toulon–Var, 2005.

[8] J. Neustupa and P. Penel, Incompressible Viscous Fluid Flows and the Generalized Impermeability Boundary Conditions, Preprint Université de Sud–Toulon–Var, 2005.

[9] R. Picard, On a Selfadjoint Realization of $\mathbf{curl}$ and Some of its Applications, *Ricerche di Matematica,* Vol. XLVII, 1998, pp. 153–180.

[10] Y. Taniuchi, On Generalized Energy Inequality of the Navier–Stokes Equations, *Manuscripta Mathematica* Vol. 94, 1997, 365–384.

[11] R. Temam, *Navier–Stokes Equations,* North–Holland Publishing Company, Amsterodam–New York–Oxford 1979.

[12] Z. Yosida and Y. Giga, Remarks on Spectra of Operator $\mathbf{rot}$, *Math. Zeitschrift,* Vol. 204, 1990, pp. 235–245.