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Hydrodynamic Model in Queueing Networks

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Abstract: - In this paper, we present a hydrodynamic model related to fluid models of high-speed communication or queueing network. We adopt a hydrodynamic analogy to describe the behavior of a fluid buffer using the Navier-Stokes equations jointly appropriate boundary conditions. The variational formulation of the problem leads to study a variational inequality. The existence of a weak solution is proved in the case where the fluid occupies the flowing domain, completely.

Key Words: - Queueing networks, Fluid buffer model, Hydrodynamic model, Variational inequality.

1 Introduction

Computer networks continue to evolve, increasing in size and complexity. There are currently several classes of models which describe in various levels of detail the behavior of the network. The utility of a particular model depends upon one' s particular goals: controlling delay, loss, stability (queue lengths remain bounded for any time). Fluid buffer models have extensively been the focus of considerable attention because of their applicability in modelling of modern communication networks. Note that, in general, the buffer content is a discrete random variable whereas in the fluid model the buffer content is a continuous random variable. Justification to considering this idealization comes from theory that establishes solidarity between idealized fluid models and more accurate discrete models, when the load is close to the capacity [2], or the state of the system is large (e.g., the network is congested [6]). Moreover fluid models lead to significant reduction in the computational effort. Indeed, when a long burst of cells or packets is sent through a link, instead of handling each individual unit, it suffices to manage only two events: the beginning of the burst and the end. Furthermore, there are several motivations for considering the deterministic approach (see [5],[7]); e.g., robust policy synthesis, i.e., any policy should be sufficiently robust to tolerate modelling errors resulting from uncertain variability in service or arrival rates. So our considerations are of deterministic type. Now we describe, briefly, the dynamics of a fluid model. Let us consider a single fluid buffer or "reservoir" of capacity $B \leq \infty$ and a work discipline service. Let $\Lambda(t) \in [0, \infty)$ be the total rate of fluid being fed into the buffer at time $t \geq 0$. The volume of the fluid arriving in the interval [0,t] is given by $A(t) = \int_0^t \Lambda(s)ds$. Let $Q(t)$ be the volume ("level") of fluid in the buffer at time $t \geq 0$. Let $R(t)$ be the output rate at $t \geq 0$. The volume of the fluid flowing out of the buffer in $[0, t]$ is $D(t) = \int_0^t R(s)ds$. The evolution of $Q(t)$ is described by

$$Q(t) = Q(0) + \int_0^t (\Lambda(s) - R(s))ds \quad (1)$$

with some allocation or control policy. For example, $s \in \{s \geq 0|(\Lambda(s) - R(t)) > 0$ or $Q(s) > 0)$ and $((\Lambda(s) - R(t) < 0$ or $Q(s) < B)\}$. Cumulative fluid losses in the interval $[0, t]$ may be computed by $L(t) = \int_0^t (\Lambda(s) - R(s))ds$ where $s \in \{s \geq 0|\Lambda(s) > R(s)$ and $Q(s) = B\}$, the set of all overflow periods. The model described so far can be applied to the more general case where the buffer is fed by N distinct fluid flows and the capacity of the buffer can be shared arbitrarily among the fluids with some control of the volume $Q_i(t)$ of i- fluid ([3], [5], [7]). We call this problem a multi-phase fluid model. It follows from the aforementioned that mathematical approach to the fluid model enables one to estimate the dynamics of the flow, delays lengths and dynamics of formation of ”queues” or ”congestion” and other ”traffic” characteristics. Moreover, it offers the advantages of being several orders of magnitude less expensive and of frequently leading to a deeper insight into the properties of the analyzed system. Furthermore the mathematical models are of significant scientific interest in connection with studies of the ”traffic” flow as physical phenomenon with complicated non trivial properties as mentioned in this introduction. The most widely used tool is the macroscopic or hydrodynamic model. In this paper we formulate a hydrodynamic model of the fluid model described above. The paper is organized as follows. In section 2 we consider a hydrodynamic model of fluid buffer model by analogy using the classical Navier-Stokes Equations. Section 4 is devoted to the variational formulation of the problem . In section 3 an outline of the existence proof of a solution to a particular case (extreme admissible state) is given.

2 Hydrodynamic model

Now we complete, from mathematical point of view, the fluid model adopted to describe a network based on hydrodynamic analogy. We assume that: Equation (1) is an integral form of the continuity equation ; The value of the divergence of the velocity of the fluid provides control of the volume of the buffer and of the inflow and outflow rate; The motion in the ”recevoir” is governed by the classical Navier-Stokes equations; The processing capacity is described by a constraint which can depend on the velocity or the volume; Dynamic conditions on the boundary simulate on/off process (in other words the boundary conditions guarantee the flowing of the fluid in the ”recevevoir”). In literature there exist several mathematical models of traffic flow (See [1], [10] for an overview) that are of some interest in mathematical modelling of networks. The description of this correlation is beyond the scope of the paper.

For simplicity of exposition we consider the flow in a cylindrical pipe Ω. The generators of the pipe are parallel to Ox_3 in the orthonormal system of axes $Ox_1x_2x_3$. Let ω be the domain of R^2 which represents a cross section of the cylinder. We denote Γ_1 the cross section in $x_3 = 0$ and Γ_2 in $x_3 = L$ (L given length of the cylinder), Γ_3 the lateral surface of the cylinder. Let n be the outward unit normal vector on $\partial\Omega$. Moreover, let $\Omega_1(t)$ and $\Omega_2(t)$ be two subsets of Ω such that

$$\Omega = \Omega_1(t) \cup \Omega_2(t),\ \Gamma(t) = (\partial\Omega_1(t) \cap \partial\Omega_2(t)) \setminus \partial\Omega.$$

Moreover the lines $\Gamma(t) \cap \Gamma_1$, $\Gamma(t) \cap \Gamma_2$ and $\Gamma(t) \cap \Gamma_3$ are not empty. We denote $\Gamma_i(t) = \Gamma_i \cap \partial\Omega_1(t)$ for $i = 1, 2, 3$.

In general $\Omega_1(t), \Omega_2(t)$ are filled with fluids of different densities and viscosities, a two-phase flow. Any way we assume that the ” fluid ” in $\Omega_2(t)$ is the vacuum (free surface problem).

In $\Omega_1(t)$ we consider the Navier-Stokes equations

$$\begin{aligned} &\partial_t u + u \cdot \nabla u - \Delta u + \nabla p = f, \\ &\nabla \cdot u = g. \end{aligned} \tag{2}$$

Here u is the velocity vector field, p the pressure, f the external force and g is a given function which can be a function of u or of the volume of $\Omega_1(t)$. The density and the viscosity are assumed constant and = 1. We complete (2) with the boundary conditions

$$\begin{aligned} &\frac{1}{2}|u|^2 + p = \alpha, \quad u \times n = 0 \ on\ \Gamma_1(t); \\ &\frac{1}{2}|u|^2 + p = \beta,\ u \times n = 0, \\ &0 \le u \cdot n < \psi,\ on\ \Gamma_2(t); \\ &u = 0\ on\ \Gamma_3(t); \\ &T(u)n = 0\ on\ \Gamma(t). \end{aligned} \tag{3}$$

Here $T = \nabla u - p$ is the stress tensor (for simplicity we consider ∇u instead of its symmetric part).

Furthermore, on $\Gamma(t)$, $u \cdot n$ represents the velocity of $\Gamma(t)$.

The conditions on $\Gamma_1(t), \Gamma_2(t)$ mean that the velocity is normal, the dynamic pressure is assigned so a constrain on the output rate is assigned. On $\Gamma_3(t)$ the classical adherence is given and on $\Gamma(t)$ the continuity condition of the normal component of the stress tensor is given in absence of surface tension.

The domain $\Omega_1(t)$ or its volume represents an admissible queue. When $\Omega = \Omega(t)$ we have the ”extreme admissible state”. The existence of a solution to (2),(3) is a complex problem. Although considerably simplified, the equations are fairly difficult to solve.

In this brief paper we consider the ”extreme admissible state” just to evaluate the difficulty of the problem.

3 Formulation in the Form of a Variational Inequality

First we give some tools from functional Analysis. In our notation we do not distinguish R-R^3

valued functions. Let Ω be a bounded open set in R^3 with boundary $\partial\Omega$ sufficiently regular. We denote $H^s(\Omega)$ the classical Sobolev space of order $s \geq 0$ on $L^2(\Omega)$. $H^0(\Omega) \equiv L^2(\Omega)$. By $((\cdot,\cdot))_s$ we will denote the scalar product in $H^s(\Omega)$ and $((\cdot,\cdot))_0 \equiv (\cdot,\cdot)$. Next, we introduce

$$\mathcal{V} = \{\phi \in C_0^\infty(\Omega), \nabla \cdot \phi = 0\},$$

V_s = closure of $\mathcal{V}$ in $H^s(\Omega)$.

V_s is a Hilbert space for the norm $||\phi||_s^2 = ((\phi,\phi))_s$ and $|\phi|_2^2 = (\phi,\phi)$. We set $V_1 = V$, V', V_s' the dual of V, V_s, respectively. $||\cdot||_1 \equiv ||\cdot||$ and $V_0 = H$. Let Ω be the cylinder defined in section 2. Moreover

$K(t)$ denotes the set of $\phi(t,x)$ for which $0 \leq \phi(x,t)\cdot n \leq \psi(x,t)$ on $\Gamma_2(t)$.

$K(t)$ is a closed convex set in $H^1(\Omega_1(t))$. Finally, $Q_T = (0,T)\times\Omega$.

Let v, u smooth functions in $K(t)$ such that $v = 0$ on Γ_3 and $v \times n = 0$ on $\Gamma_1 \cup \Gamma_2$. Multiplying $(2)_1$ by $v-u$ and after integration by parts on $\Omega_1(t)$, having in mind the boundary conditions (3), we get

$$\int_0^T ((\partial_t v, v-u)_{\Omega_1(t)} + (\nabla u, \nabla(v-u))_{\Omega_1(t)} - \frac{1}{2}(v,|u|^2 n)_{\Gamma_1(t)\cup\Gamma_2(t)} + (\alpha n, v-u)_{\Gamma_1(t)} + (\beta n, v-u)_{\Gamma_2(t)} + \frac{1}{2}(u, n|v|_2^2)_{\Gamma(t)} - (u\cdot nu, v)_{\Gamma(t)} + (u\cdot\nabla u, v)_{\Omega_1(t)} - (p, \nabla\cdot(v-u)))dt \geq -\frac{1}{2}|v_0 - u_0|_2^2. \quad (4)$$

The existence of a weak solution to 3D Navier-Stokes equations in a time-independent convex set was solved by the author in [8] (see [4] for a discussion of the problem). The case where the convex set depends on time the existence is an open problem. In [9] some particular cases are considered. Moreover a variational inequality related to the free surface problem is a new problem, completely.

In this brief paper we intend to consider the case where $\Omega \equiv \Omega(t)$ for any t -the extreme admissible queue problem described in section 2 - with $g = 0$ to give evidence about the complexity of the problem for a "simplified" case.

We introduce ($|\cdot| =$ absolute value)

$$\mathcal{W} = \{\phi|\phi \in C^\infty(\Omega),\ |\phi\cdot n| = |\phi| \ on\ \Gamma_1\cup\Gamma_2,$$

$$\phi = 0\ on\ \Gamma_3,\ \nabla\cdot\phi = 0\},$$

W = closure of $\mathcal{W}$ in $H^1(\Omega)$, W' the dual of W.

We notice that, for $\phi = (\phi_1,\phi_2,\phi_3)$ in W, on $\Gamma_1\cup\Gamma_2$, $\partial_{x_1}\phi_1 = \partial_{x_2}\phi_2 = 0$ consequently $\partial_{x_3}\phi_3 = 0$ thanks to $\nabla\cdot\phi = 0$. We set $\phi^+ = (sup(\phi_i, 0))$, $\phi^- = (inf(\phi_i,0))$ for $i = 1,2,3$. Moreover the Friedricks-Poncarè inequality holds.

We prove the following

Theorem - *Let $\alpha, \beta \in L^2(0,T; H^{-1/2}(\Gamma_1\cup\Gamma_2)$, $0 < c \leq \psi \in L^2(0,T;L^2(\Gamma_2))$ and $u_0 \in L^2(\Omega)\cap K(t)$ and $\nabla\cdot u_0 = 0$. Then there exists a (weak solution) $u \in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;W)$ such that the inequality (4) is satisfied for any $v \in L^2(0,T;W\cap K(t))\cap L^6(Q_t)$, $\partial_t v \in L^2(0,T;W')$.*

4 Outline of the Proof.

Step 1 - Penalized System. We consider the following approximate problem (for simplicity we set $\alpha = \beta, u^{\epsilon,\eta} \equiv u$)

$$\int_0^T \eta(\partial_t u, \partial_t\phi) - (u, \partial_t\phi) + (\nabla u, \nabla\phi) + (u\cdot\nabla\bar{u}, \bar{\phi}) + (\alpha n - \frac{1}{2}|\bar{u}|^2 n, \phi)_{\Gamma_1\cup\Gamma_2} + \frac{1}{\epsilon}[((u\cdot n-\psi)^+_{\Gamma_2}, \phi) + ((u\cdot n)^-, \phi)_{\Gamma_2}])dt + (u(T),\phi(T)) = \int_0^T (f,\phi)dt + (u_0, \phi(0)). \quad (5)$$

Here η, ϵ are positive parameters and $\bar{u}, \bar{\phi}$ are regularizations of u, ϕ dependent on parameter ϵ.

We denote by $a(u,\phi)$ the left-hand side in (5). By direct computation, we have $a(u,u) \geq c||u||_{H^1(Q_T)}$ and obtain the existence of a solution in $H^1(Q_T)\cap W$ of (5). To pass to the limit in (5) we will need a priori estimates of the approximations u.

Step 2. We replace ϕ with u in (5). After some calculations, we have

$$\eta\int_0^T |\partial_t u|_2^2 dt \leq c,\ \int_0^T \|u\|^2 dt \leq c, \int_0^T (\int_{\Gamma_2} (|(u\cdot n-\psi)^+|^2 + |(u\cdot n)^-|^2)d\Gamma)dt \leq c\epsilon, |u(T)|_2^2 \leq c,\ |u(0)|_2^2 \leq c. \quad (6)$$

It follows that $u^{\eta,\epsilon} \rightarrow u^\epsilon$ weakly in $L^2(0,T;W)$. Then passing to the limit with respect to η in (5) with smooth ϕ with support in $(0,T)$ we obtain

$$\int_0^T (-(u,\partial_t\phi) + (\nabla u,\nabla\phi) + (u\cdot\nabla\bar{u},\bar{\phi}) + (\alpha - \frac{1}{2}|\bar{u}|^2,\phi)_{\Gamma_1\cup\Gamma_2} + \frac{1}{\epsilon}((u\cdot n-\psi)^+,\phi)_{\Gamma_2} + ((u\cdot n)^-,\phi)_{\Gamma_2}) - (f,\phi))dt. \quad (7)$$

To pass to the limit with respect to ϵ we need the convergence in $L^2(Q_T)$ (for example).

Step 3. In (7) we consider $\phi(t)\in C_0^1(0,T;\mathcal{V})$ and obtain

$$\int_0^T (u,\partial_t\phi)dt = \int_0^T ((\nabla u,\nabla\phi) + (u\cdot\nabla\bar{u},\bar{\phi}) - (f,\phi))dt \quad (8)$$

(8) implies that $\partial_t u^\epsilon$ is bounded in $L^2(0,T;V_2')$ uniformly with respect to ϵ. Using the classical compactness result (see [4]) we obtain $\{Pu^\epsilon\}$ is a compact set in $L^2(Q_T)$ where P is the projector operator from $L^2(\Omega)$ onto H. Now we prove that $\{u^\epsilon\}$ converges in $L^2(Q_T)$. Let Ω_δ be the cylinder contained in Ω with $\delta\le x_3\le L-\delta$ and cross section ω. Let $\theta_\delta\equiv\theta\in C_0^\infty(\Omega)$, Supp$\theta\subset\Omega$, $\theta\equiv 1$ on Ω_δ. Let $w_\delta^\epsilon\equiv w$ be the solution in $\tilde{\Omega}=\Omega\setminus\Omega_\delta$ of

$$\nabla\cdot w = u^\epsilon\nabla\theta,\ w=0\ on\ \partial\tilde{\Omega}.$$

We notice that

$$\int_0^T\int_{\tilde{\Omega}}|u^\epsilon|^2dxdt \le$$

$$c\delta^{2/3}\int_0^T(\int_{\tilde{\Omega}}|u^\epsilon|^6dx)^{1/3}dt \le c\delta^{2/3}.$$

We set $h^\epsilon = \theta(u^\epsilon - u) - w$, (w is extended by 0 in Ω_δ). Note that $\nabla\cdot h^\epsilon = 0$ and $h^\epsilon\in L^2(0,T;H_0^1(\Omega))$.

Now

$$\int_0^T\int_{\Omega_\delta}h^\epsilon u^\epsilon dxdt =$$

$$\int_0^T(\int_\Omega h^\epsilon u^\epsilon dx - \int_{\tilde{\Omega}}h^\epsilon u^\epsilon dx)dt.$$

The first integral on the right-hand side tends to zero as $\epsilon\to 0$ because of $h^\epsilon\to 0$ weakly in $L^2(0,T;H_0^1(\Omega))$ and $Pu^\epsilon\to Pu$ strongly in $L^2(0,T;H^{-1}(\Omega))$. Moreover the second integral satisfies

$$|\int_0^T\int_{\tilde{\Omega}}h^\epsilon u^\epsilon dxdt| \le c\delta^{2/3}$$

with c independent of ϵ. The strong convergence in $L^2(Q_T)$ of $\{u^\epsilon\}$is proved.

An alternative proof of the strong convergence of $\{u^\epsilon\}$ can be obtained by the use of the compactness theorem of Fr$\acute{e}$chet-Kolmogorov. In fact, consider the Steklov function

$$u_h^\epsilon = \int_{t-h}^t u^\epsilon(x,s)ds.$$

Here $h>0$ and $u^\epsilon(x,t)$ can be considered 0 for $t<0$.

By virtue of the estimates (6) and the Jensen inequality one has

$$\int_0^T(\partial_t u^\epsilon,u_h^\epsilon)dt = (u^\epsilon(T),\frac{1}{h}\int_{T-h}^T u^\epsilon(s,x)) - \frac{1}{h}\int_0^T(u^\epsilon(t),u^\epsilon(t)-u^\epsilon(t-h))dt \le$$

$$\frac{c}{\sqrt{h}} - \frac{1}{2h}\int_0^T|u^\epsilon(t)-u^\epsilon(t-h|_2^2dt;$$

$$|\int_0^T(\nabla u^\epsilon,\nabla u_h^\epsilon)dt| \le$$

$$\int_0^T\|u^\epsilon\|^2\frac{1}{\sqrt{h}}(\int_{t-h}^t\|u^\epsilon\|^2ds)^{1/2}dt \le \frac{c}{\sqrt{h}};$$

$$|\int_0^T(u^\epsilon\cdot\nabla\bar{u}^\epsilon,\bar{u}_h^\epsilon)dt| \le \frac{c}{\sqrt{h}};$$

$$\frac{1}{\epsilon}\int_0^T((u^\epsilon\cdot n)^-,u_h^\epsilon)_{\Gamma_2}dt =\le \frac{c}{\sqrt{h}};$$

$$\frac{1}{\epsilon}\int_0^T((u^\epsilon\cdot n-\psi)^+,u_h^\epsilon)_{\Gamma_2}dt \le \frac{1}{\sqrt{h}}.$$

In (7) we replace ϕ with u_h^ϵ and thanks to the above estimates we obtain

$$\int_0^T|u^\epsilon(t)-u^\epsilon(t-h)|_2^2dt \le c\sqrt{h}. \quad (9)$$

The estimates on ∇u^ϵ and (9) imply the compactness of $\{u^\epsilon\}$ in $L^2(Q_T)$. Now it is routine matter to obtain (4).

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