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The Navier–Stokes Equation for Incompressible Fluid and the Generalized Impermeability Boundary Conditions

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Abstract: - A theoretical foundation of the generalized impermeability boundary conditions for the Navier–Stokes equations is given in [1]. Although main results of the classical theory, already known for the Dirichlet boundary condition, are true for the generalized impermeability boundary conditions as well, one can also prove some finer theorems. In this brief article, we review some results from [1] and bring additional results and comments.

Key-Words: - Navier–Stokes equations, Boundary conditions

1 Natural Question of Natural Boundary Conditions

In most works on incompressible viscous fluid flows, the Navier-Stokes equation

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \nu \Delta \boldsymbol{u} + \boldsymbol{f} \tag{1}$$

and the equation of continuity

$$\operatorname{div} \boldsymbol{u} = 0 \tag{2}$$

are treated with the homogeneous Dirichlet boundary condition

$$\boldsymbol{u} = \boldsymbol{0} \tag{3}$$

on the part of the boundary which coincides with a fixed wall. However, physicists and engineers know that this condition not always well reflects the behavior of the fluid on and near the boundary and it is true especially if the boundary is smooth and the viscosity of the fluid is small. (3) is equivalent with the two conditions $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ and $\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0}$ where $\boldsymbol{n}$ is the outer normal vector on the boundary. The first equation expresses the zero flux through the boundary. The second equation says that the tangential component of the velocity $\boldsymbol{u}$ is zero on the boundary and it is precisely this part which expresses the no–slip boundary condition and which is a matter of discussion.

In this paper, we consider the boundary conditions

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0, \ \ \mathbf{curl}\, \boldsymbol{u} \cdot \boldsymbol{n} = 0, \ \ \mathbf{curl}^2 \boldsymbol{u} \cdot \boldsymbol{n} = 0. \tag{4}$$

We call them the *generalized impermeability boundary conditions.*

One of the questions which immediately arise is the question of a physical sense of the boundary conditions (4). The first condition in (4) coincides with the first part of (3) and we already know that it expresses the zero flux through the boundary. The second condition in (4) requires the normal component of vorticity to be zero on the boundary. This condition is also contained in (3) – see Section 3, Lemma 2. Since $\mathbf{curl}^2 \boldsymbol{u} = -\Delta \boldsymbol{u}$ for a divergence vector field $\boldsymbol{u}$, the third condition in (4) can be rewritten in the form $\nu \Delta \boldsymbol{u} \cdot \boldsymbol{n} = \operatorname{Div} \mathbb{T}_D \cdot \boldsymbol{n} = 0$ where $\mathbb{T}_D$ is the dynamic stress tensor and $\operatorname{Div} \mathbb{T}_D$ represents the vector of intensity of the local source of tensor field $\mathbb{T}_D$. The third condition in (4) expresses the requirement that the normal component of this vector equals zero on the boundary. Lemma 2 in Section 3 shows that in fact, the third condition in (4) is the only point where the boundary conditions (3) and (4) differ.

2 Elements from the Theory of Operator curl

Assume that the considered fluid fills a bounded simply connected domain $\Omega \subset \mathbb{R}^3$ whose boundary $\partial\Omega$ is a smooth surface. We shall use the notation:

- $L^2_\sigma(\Omega)^3$ is a subspace of $L^2(\Omega)^3$ which contains functions $\boldsymbol{u}$ whose divergence equals zero in Ω in the sense of distributions and $(\boldsymbol{u} \cdot \boldsymbol{n})|_{\partial\Omega} = 0$ in the sense of traces.

- P_σ is the orthogonal projection of $L^2(\Omega)^3$ onto $L^2_\sigma(\Omega)^3$. Q_σ is the complementary projection, i.e. $Q_\sigma = I - P_\sigma$.
- D^1 is the set of functions $\boldsymbol{u} \in W^{1,2}(\Omega)^3 \cap L^2_\sigma(\Omega)^3$ such that $(\mathbf{curl}\,\boldsymbol{u}\cdot\boldsymbol{n})|_{\partial\Omega} = 0$ in the sense of traces. It is a closed subspace of $W^{1,2}(\Omega)^3$.
- D^{-1} is the dual to D^1. The duality between the elements of D^{-1} and D^1 is denoted by $\langle\, .\,,\, .\rangle$.
- $A = \mathbf{curl}\,|_{D^1}$ (Thus, $D^1 = D(A)$.)
- $D^2 = D(A^2)$. It is shown in [1] that it is the set of functions $\boldsymbol{v} \in W^{2,2}(\Omega)^3 \cap D^1$ such that $(\mathbf{curl}^2\boldsymbol{v}\cdot\boldsymbol{n})|_{\partial\Omega} = 0$ in the sense of traces.
- $\sigma(A)$ (respectively $\rho(A)$) is the spectrum (respectively the resolvent set) of operator A, as an operator in $L^2_\sigma(\Omega)^3$.
- $\|\,.\,\|_2$ denotes the norm in $L^2_\sigma(\Omega)^3$ and $\|\,.\,\|_{k,2}$ denotes the norm in $W^{k,2}(\Omega)^3$.

It is well known that the orthogonal complement of $L^2_\sigma(\Omega)^3$ in $L^2(\Omega)^3$ is the space of gradients $\nabla\varphi$ such that $\varphi \in W^{1,2}(\Omega)$. D^1 is dense in $L^2_\sigma(\Omega)^3$ and A maps D^1 onto $L^2_\sigma(\Omega)^3$. We cite several results whose proofs can be found in [1]. (We also refer to R. Picard [11] for the proof of part c).)

Lemma 1 *a) D^1 equals the set of functions of the form $\boldsymbol{v} = \boldsymbol{v}_0 + \nabla\varphi$ where $\boldsymbol{v}_0 \in W^{1,2}_0(\Omega)^3$, $\Delta\varphi = -\mathrm{div}\,\boldsymbol{v}_0$ in Ω and $\partial\varphi/\partial\boldsymbol{n}\,|_{\partial\Omega} = 0$.*
b) $D^1 = P_\sigma\, W^{1,2}_0(\Omega)^3$
c) Operator A is selfadjoint in $L^2_\sigma(\Omega)^3$ and its resolvent operator is compact in $L^2_\sigma(\Omega)^3$ for all $\lambda \in \rho(A)$.
d) $\sigma(A) = \{\lambda_i;\ i \in \mathbb{Z}^*\}$ ($\mathbb{Z}^* = \mathbb{Z} - \{0\}$) where λ_i are isolated real eigenvalues with the same finite algebraic and geometric multiplicity which cluster at $+\infty$ and $-\infty$.
e) $\|A^k\,.\,\|_2$ represents the norm in D^k, equivalent with the norm $\|\,.\,\|_{k,2}$ for $k = 1, 2$.

3 An Equivalent Form of the Dirichlet Boundary Condition (3)

Lemma 2 *A function $\boldsymbol{u} \in W^{1,2}(\Omega)^3 \cap L^2_\sigma(\Omega)^3$ satisfies the homogeneous Dirichlet boundary condition (3) if and only if it satisfies*

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0, \quad \mathbf{curl}\,\boldsymbol{u}\cdot\boldsymbol{n} = 0, \quad \frac{\partial\boldsymbol{u}}{\partial\boldsymbol{n}}\cdot\boldsymbol{n} = 0 \qquad (5)$$

on the boundary $\partial\Omega$ of domain Ω.

Proof: Assume that $\boldsymbol{u}$ is a smooth function in $L^2_\sigma(\Omega)^3$ at first.

If $\boldsymbol{u}$ satisfies (3) then $\boldsymbol{u}$ and $\mathbf{curl}\,\boldsymbol{u}$ obviously satisfy the first two conditions in (5). Let us verify the third condition. Let $\boldsymbol{x}_0 \in \partial\Omega$. The cartesian system of coordinates can be chosen so that the origin is at point $\boldsymbol{x}_0$ and $\boldsymbol{n}$ shows the direction of the x_3–axis. Since $u_1 = u_2 = 0$ on $\partial\Omega$ and ∂_1, ∂_2 represent tangential derivatives at point $\boldsymbol{x}_0$, we have $\partial_1 u_1 + \partial_2 u_2 = 0$ at $\boldsymbol{x}_0$. This implies, due to the equation of continuity (2), that $\partial_3 u_3 = 0$ at $\boldsymbol{x}_0$. This equation is identical with the third condition in (5) at point $\boldsymbol{x}_0$.

On the other hand, let $\boldsymbol{u}$ satisfy (5). The third condition in (5) implies that $\boldsymbol{u}$ satisfies the two–dimensional surface form of the equation of continuity (2) on $\partial\Omega$, which means that if C is a closed simple smooth curve on $\partial\Omega$ then the flux through C on $\partial\Omega$ equals zero. This can be expressed by the formula

$$\oint_C \boldsymbol{u}\cdot(\mathrm{d}\boldsymbol{l}\times\boldsymbol{n}) = 0. \qquad (6)$$

Due to the first two conditions in (5), $\boldsymbol{u}$ belongs to space D^1. Lemma 1 and the smoothness of $\boldsymbol{u}$ imply that $\boldsymbol{u}$ coincides with $\nabla\varphi$ (for some $\varphi \in W^{3,2}(\Omega)$) on $\partial\Omega$. If function φ is not constant on $\partial\Omega$ then it has a maximum on $\partial\Omega$ at some point $\boldsymbol{y} \in \partial\Omega$ and there exists a closed simple smooth curve C around point $\boldsymbol{y}$ on $\partial\Omega$ such that $\nabla\varphi$ differs from the zero vector on a part of C which has a positive 1–dimensional measure, $\nabla\varphi$ is perpendicular to C and shows to the "interior" of C in all points of curve C. Then $\nabla\varphi\times\boldsymbol{n}$ is tangent to C and we can suppose that C is oriented by this tangent vector. This implies that

$$\oint_C \boldsymbol{u}\cdot(\mathrm{d}\boldsymbol{l}\times\boldsymbol{n}) = \oint_C (\boldsymbol{u}\times\boldsymbol{n})\cdot\mathrm{d}\boldsymbol{l} = \oint_C (\nabla\varphi\times\boldsymbol{n})\cdot\mathrm{d}\boldsymbol{l} \,>\, 0$$

which is in contradiction with (6). Thus, φ is a constant function on $\partial\Omega$ and the tangent components of $\nabla\varphi$ on $\partial\Omega$ (which coincide with the tangent components of $\boldsymbol{u}$) equal zero. This confirms that $\boldsymbol{u}$ satisfies the homogeneous Dirichlet boundary condition (3).

The statement of the lemma can finally be extended to $\boldsymbol{u} \in W^{1,2}(\Omega)^3 \cap L^2_\sigma(\Omega)^3$ by means of the density argument. The third condition in (5) is satisfied in the sense of duality between elements of $W^{-1/2,2}(\partial\Omega)$ and $W^{1/2,2}(\partial\Omega)$: $\langle\partial_j\boldsymbol{u}\cdot\boldsymbol{n},\, n_j\rangle_{\partial\Omega} = 0$. Indeed, $\partial_j\boldsymbol{u} \in L^2(\Omega)^3$ (for $j = 1, 2, 3$) and its divergence equals zero in the sense of distributions, hence the trace of $\partial_j\boldsymbol{u}\cdot\boldsymbol{n}$ on $\partial\Omega$ belongs to $W^{-1/2,2}(\partial\Omega)$. □

Lemma 2 confirms that the generalized impermeability boundary conditions (4) differ from the no–slip boundary condition (3) only in the third condition in (4) and (5). We shall see that this difference has interesting consequences.

4 Boundary Conditions for Vorticity and Pressure

Assume, for simplicity, that $\boldsymbol{f} = \boldsymbol{0}$. Denote $\boldsymbol{\omega} = \mathbf{curl}\,\boldsymbol{u}$. Applying operator $\mathbf{curl}$ to the Navier–Stokes equation (1), we obtain the equation

$$\partial_t \boldsymbol{\omega} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u} = \nu\Delta\boldsymbol{\omega}. \qquad (7)$$

If the Navier–Stokes equation is considered with the homogeneous Dirichlet boundary condition (3) then we can only derive that the normal component of $\boldsymbol{\omega}$ equals zero on the boundary, but this information is not sufficient in order to formulate a well–posed boundary–value problem for function $\boldsymbol{\omega}$, based on equation (7). We are going to show that if $\boldsymbol{u}$ satisfies the boundary conditions (4) on $\partial\Omega$ then $\boldsymbol{\omega}$ (if it is smooth enough) also satisfies the boundary conditions (4), i.e.

$$\boldsymbol{\omega}\cdot\boldsymbol{n} = 0, \;\; \mathbf{curl}\,\boldsymbol{\omega}\cdot\boldsymbol{n} = 0, \;\; \mathbf{curl}^2\boldsymbol{\omega}\cdot\boldsymbol{n} = 0 \qquad (8)$$

on $\partial\Omega$. The first two conditions in (8) directly follow from (4). Thus, we only need to show that $\boldsymbol{\omega}$ satisfies the third condition in (8). Since $\nu\Delta\boldsymbol{\omega} = -\nu\,\mathbf{curl}^2\boldsymbol{\omega}$ in equation (7), it is sufficient to show that

$$[\,(\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u}\,]\cdot\boldsymbol{n} = 0 \qquad (9)$$

on $\partial\Omega$. However,

$$(\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u} \;=\; \mathbf{curl}(\boldsymbol{\omega}\times\boldsymbol{u}).$$

Both $\boldsymbol{\omega}$ and $\boldsymbol{u}$ are tangent to $\partial\Omega$. Thus, their cross product is normal to $\partial\Omega$ and its $\mathbf{curl}$ is again tangent. This implies (9).

Applying operator div to the Navier–Stokes equation (1), we obtain the well known Poisson–type equation for pressure:

$$\Delta p \;=\; -\partial_i\partial_j\,(u_i\,u_j) + \operatorname{div}\boldsymbol{f}. \qquad (10)$$

Equation (10) can be solved with the Neumann boundary condition

$$\frac{\partial p}{\partial \boldsymbol{n}} \;=\; \nu\Delta\boldsymbol{u}\cdot\boldsymbol{n} - [(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]\cdot\boldsymbol{n} \qquad (11)$$

on $\partial\Omega$ which directly follows from equation (1). Applying projection Q_σ to equation (1), we obtain $\nabla p = \nabla p^I + \nabla p^{II}$ where $\nabla p^I = \nu Q_\sigma\Delta\boldsymbol{u}$ and $\nabla p^{II} = Q_\sigma[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]$. Function p^I is harmonic. In the case of boundary conditions (4), projections P_σ and Q_σ commute with Δ because $\Delta = -A^2$. Hence $\nabla p^I = \nu\Delta Q_\sigma\boldsymbol{u} = \boldsymbol{0}$ and p^I can be taken to be equal to zero. Furthermore, $\nu\Delta\boldsymbol{u}\cdot\boldsymbol{n} = \nu\Delta P_\sigma\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$ and thus the first term on the right hand side of (11) can be omitted.

Boundary conditions (4) also enable to derive a Dirichlet–type boundary condition for pressure. Suppose that $\boldsymbol{u}$ is a smooth solution of (1), (2) which satisfies boundary conditions (4) and $\boldsymbol{f}(\,.\,,t) \in D^1$ for all (or almost all) t from the considered time interval. Then, due to Lemma 1, $\boldsymbol{u} = \boldsymbol{u}_0 + \nabla\varphi$ where $\boldsymbol{u}_0 = \boldsymbol{0}$ on $\partial\Omega$. By analogy, $\boldsymbol{f} = \boldsymbol{f}_0 + \nabla\chi$ where $\boldsymbol{f}_0 = \boldsymbol{0}$ on $\partial\Omega$. Finally, the identity $\mathbf{curl}^2\boldsymbol{\omega}\cdot\boldsymbol{n} = 0$ in (8) means that $\mathbf{curl}\,A^2\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$ and so $A^2\boldsymbol{u}$ belongs to D^1, too. Hence $A^2\boldsymbol{u} = \boldsymbol{w}_0 + \nabla\psi$ where $\boldsymbol{w}_0(x,t) = 0$ if $x \in \partial\Omega$. The Navier–Stokes equation (1) can be written in the form

$$\partial_t\boldsymbol{u} + A\boldsymbol{u}\times\boldsymbol{u} = -\nabla q - \nu A^2\boldsymbol{u} + \boldsymbol{f} \qquad (12)$$

where $q = p + \frac{1}{2}\,|\boldsymbol{u}|^2$. The tangential component of the term $A\boldsymbol{u}\times\boldsymbol{u}$ on $\partial\Omega$ is zero because it is the cross product of the two tangential vectors on $\partial\Omega$. Thus, assuming that equation (12) is satisfied up to the boundary and multiplying it by an arbitrary tangent vector $\boldsymbol{\tau}$, we can obtain

$$\nabla\left(\partial_t\varphi + \nu\psi - \chi + p + \tfrac{1}{2}\,|\nabla\varphi|^2\right)\cdot\boldsymbol{\tau} = 0.$$

This implies that

$$\partial_t\varphi + \nu\psi - \chi + p + \tfrac{1}{2}\,|\nabla\varphi|^2 \;=\; h(t) \qquad (13)$$

on $\partial\Omega$ where h is a function of time. Using the fact that pressure p is determined uniquely up to an additive function of time, we can choose $h(t) = 0$. Equation (13) suggests the Dirichlet boundary condition for p, of course in a situation when the information on the other quantities φ, ψ and χ can be obtained separately.

5 The Weak Problem with Boundary Conditions (4)

The initial–boundary value problem for equations (12), (2) with the initial condition $\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0$ and with the generalized impermeability boundary conditions (4) can be weakly formulated in this way: *Let $T > 0$, $\boldsymbol{f} \in L^2(0,T;\, D^{-1})$ and $\boldsymbol{u}_0 \in L^2_\sigma(\Omega)^3$. Denote $Q_T = \Omega\times(0,T)$. We look for $\boldsymbol{u} \in L^\infty(0,T;\, L^2_\sigma(\Omega)^3)\cap L^2(0,T;\, D^1)$ such that*

$$\int_{Q_T}\Big(-\boldsymbol{u}\cdot\partial_t\boldsymbol{\phi} + (A\boldsymbol{u}\times\boldsymbol{u})\cdot\boldsymbol{\phi} + A\boldsymbol{u}\cdot A\boldsymbol{\phi}\Big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$
$$-\int_\Omega \boldsymbol{u}_0\cdot\boldsymbol{\phi}(\,.\,,0)\,\mathrm{d}\boldsymbol{x} = \int_0^T \langle \boldsymbol{f}(\,.\,,t), \boldsymbol{\phi}(\,.\,,t)\rangle\,\mathrm{d}t$$

for all $\boldsymbol{\phi} \in C^\infty([0,T];\, D^1)$ such that $\boldsymbol{\phi}(\,.\,,T) = \boldsymbol{0}$. We shall denote this weak problem by (WP). It can be shown that if $\boldsymbol{u}$ (together with pressure p) is a strong solution to the problem (1), (2), (4) with the initial condition $\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0$ then $\boldsymbol{u}$ is a weak solution. In order to confirm the sense of the weak formulation, it is also necessary to show the opposite, i.e. that *to a sufficiently smooth weak solution $\boldsymbol{u}$ there exists an associated pressure p so that $\boldsymbol{u}$, p is a strong solution.* The most steps of the proof are standard. Using at first test functions $\boldsymbol{\phi}$ that have for each $t \in [0,T]$ a compact support in Ω, we can show that there exists a smooth function q such that the pair $\boldsymbol{u}$, q satisfies equations (12) and (2) together with the initial condition $\boldsymbol{u}(\,.\,,0) = \boldsymbol{u}_0$ in a strong sense. The validity of the first two conditions in (4) directly follows from the fact that $\boldsymbol{u}(\,.\,,t) \in D^1$ for $t \in (0,T)$. The crucial part is to prove that $\boldsymbol{u}$ satisfies in the sense of traces the third boundary condition $\mathbf{curl}^2\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on $\partial\Omega \times (0,T)$. It is not obvious because a weak solution does not necessarily belong to the domain of $\mathbf{curl}^2$. Nevertheless, choosing a general test function $\boldsymbol{\phi}$, integrating by parts in the integral identity in the weak formulation and using the information that $\boldsymbol{u}$ is a strong solution, we obtain:

$$\int_0^T \int_{\partial\Omega} \mathbf{curl}\,\boldsymbol{u}\cdot(\boldsymbol{\phi}\times\boldsymbol{n})\,\mathrm{d}S\,\mathrm{d}t = 0. \tag{14}$$

Function $\boldsymbol{\phi}$ can be, in accordance with Lemma 1, expressed in the form $\boldsymbol{\phi} = \boldsymbol{\phi}_0 + \nabla\varphi$ where $\boldsymbol{\phi}_0 \in C^\infty([0,T];\, W_0^{1,2}(\Omega)^3)$ and φ is, for each $t \in [0,T]$, a solution of the Neumann problem

$$\Delta\varphi = -\nabla\cdot\boldsymbol{\phi}_0 \ \text{ in } \Omega, \qquad \frac{\partial\varphi}{\partial\boldsymbol{n}}\bigg|_{\partial\Omega} = 0$$

Substituting $\boldsymbol{\phi} = \boldsymbol{\phi}_0 + \nabla\varphi$ into (14), we obtain:

$$\begin{aligned} 0 &= \int_0^T \int_{\partial\Omega} \mathbf{curl}\,\boldsymbol{u}\cdot(\nabla\varphi\times\boldsymbol{n})\,\mathrm{d}S\,\mathrm{d}t \\ &= -\int_0^T \int_{\Omega} \mathrm{div}\,(\nabla\varphi\times\mathbf{curl}\,\boldsymbol{u})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ &= \int_0^T \int_{\Omega} \nabla\varphi\cdot\mathbf{curl}^2\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ &= \int_0^T \langle(\mathbf{curl}^2\boldsymbol{u}\cdot\boldsymbol{n}),\,\varphi\rangle_{\partial\Omega}\,\mathrm{d}t. \end{aligned} \tag{15}$$

$\mathbf{curl}^2\boldsymbol{u}$, for a.a $t \in (0,T)$, is a divergence–free function in $L^2(\Omega)^3$ and so its normal component on the boundary belongs to $W^{-1/2,2}(\partial\Omega)$. The term $\langle(\mathbf{curl}^2\boldsymbol{u}\cdot\boldsymbol{n}),\,\varphi\rangle_{\partial\Omega}$ therefore expresses the duality between the elements of $W^{-1/2,2}(\partial\Omega)$ and $W^{1/2,2}(\partial\Omega)$. The set of traces on $\partial\Omega$ of all possible functions φ is dense in $W^{1/2,2}(\partial\Omega)$. Thus, (15) implies that for a.a. $t \in (0,T)$, $\mathbf{curl}^2\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$ in the sense of traces. This shows that the third boundary condition in (4) is implicitly contained in the formulation of the weak problem (WP).

If $\boldsymbol{u}$ is a weak solution, i.e. a solution of problem (WP), then we can prove similarly as in the case of the Dirichlet boundary condition (3) that there exists an associated pressure p such that $\boldsymbol{u}$ and q (where $q = p + \frac{1}{2}|\boldsymbol{u}|^2$) satisfy equation (12) in the sense of distributions in Q_T.

Boundary conditions (4) enable to derive many results which are already known to hold for the Navier–Stokes equation with the homogeneous Dirichlet boundary condition. We present some of them.

Theorem 3 (Global in time existence of a weak solution) *The weak problem (WP) has a solution $\boldsymbol{u}$ which satisfies the strong energy inequality*

$$\begin{aligned} &\|\boldsymbol{u}(\,.\,,t)\|_2^2 + 2\nu\int_\xi^t \|\nabla\boldsymbol{u}(\,.\,,\xi)\|_2^2\,\mathrm{d}\xi \\ &\quad\le\ 2\int_\xi^t \langle\boldsymbol{f}(\,.\,,\sigma),\boldsymbol{u}(\,.\,,\sigma)\rangle\,\mathrm{d}\sigma + \|\boldsymbol{u}(\,.\,,\xi)\|_2^2 \end{aligned} \tag{16}$$

for a.a. $\xi \in (0,T)$ and all $t \in [\xi,T)$ and $\lim_{t\to0+}\|\boldsymbol{u}(\,.\,,t)-\boldsymbol{u}_0\|_2 = 0$. Moreover, if $\boldsymbol{u}_0 \in D^1$ then the weak solution $\boldsymbol{u}$ can be constructed so that it satisfies

$$\int_0^T \|\boldsymbol{u}\|_{2,2}^{2/3}\,\mathrm{d}t\ <\ +\infty. \tag{17}$$

The theorem can be proved by the Galerkin method. The idea is due to J. Leray [9] and E. Hopf [7] and it can also be used in the case of the generalized impermeability boundary conditions (4). The approximations can naturally be constructed as linear combinations of eigenfunctions of operator A.

The energy inequality was originally proved by J. Leray (in $\mathbb{R}^3$) and by E. Hopf (in abounded domain in $\mathbb{R}^3$ with the Dirichlet boundary condition (3)) only for $\xi = 0$. The generalization for a.a. $\xi \in (0,T)$, which is possible in the case of a large class of domains Ω, was discovered later and the inequality has therefore been called the “strong energy inequality”.

Estimate (17) was proved by C. Foias, C. Guillope and R. Temam [5] in the space–periodic case in $\mathbb{R}^3$ and it was later modified by G. F. D. Duff [4] for the case of a bounded domain Ω with the Dirichlet boundary condition (3). In our case, the integrability of $\|A^2\boldsymbol{u}_n\|_2^{2/3}$ on $(0,T)$ can be at first established for approximations $\boldsymbol{u}_n$ which are linear combinations of eigenfunctions of operator A. Estimate (17) can be obtained by a usual limit procedure for $n \to +\infty$. (17) provides another explication of the mathematical sense in which the weak solution satisfies the third

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condition in (4): it shows that $\boldsymbol{u}(\,.\,,t) \in W^{2,2}(\Omega)^3$ for a.a. $t \in (0,T)$. So $\mathbf{curl}^2\boldsymbol{u}(\,.\,,t) \in L^2(\Omega)^3$ and since it is divergence–free, its normal component on the boundary belongs to $W^{-1/2,2}(\partial\Omega)$.

Let us further note that the generalized impermeability boundary conditions (4) enable to prove uniqueness in the same well known classes of weak solutions (defined by the so called Prodi–Serrin integrability conditions) as the Dirichlet boundary condition (3). Indeed, following the procedures described e.g. by G. P. Galdi [6], one can verify that they can also be performed, with only minor modifications, considering boundary conditions (4).

6 More from the Theory of the Navier–Stokes Equation with Boundary Conditions (4)

Suppose further for simplicity that $\boldsymbol{f} = \boldsymbol{0}$. It can be easily observed that if solution $\boldsymbol{u}$ of the Navier–Stokes equation satisfies the boundary conditions (4) then $P_\sigma\Delta\boldsymbol{u} = \Delta P_\sigma\boldsymbol{u} = \Delta\boldsymbol{u} = A^2\boldsymbol{u}$. The fact that P_σ commutes with the Laplace operator has important consequences. It enables, except others, to improve some fine results from the theory of the Navier–Stokes equation. Let us mention at least two of them:

I. (Interior regularity of a weak solution) The classical results of J. Serrin [12] say that if $\boldsymbol{u}$ is a weak solution to the Navier–Stokes equation that satisfies inequality (16) and at least one of the conditions

(i) $\boldsymbol{u} \in L^s(t_1,t_2;\, L^r(\Omega_1)^3)$ *for some* r, s *such that* $2 \le s \le +\infty,\ 3 < r \le +\infty,\ 2/s + 3/r = 1,$

(ii) the norm of $\boldsymbol{u}$ *in* $L^\infty(t_1,t_2;\, L^3(\Omega_1)^3)$ *is sufficiently small*

(Ω_1 is a sub–domain of Ω and $t_1 < t_1+\zeta < t_2-\zeta < t_2$) then $\boldsymbol{u}$ has all space derivatives in $L^\infty(t_1+\zeta, t_2-\zeta;\, L^\infty(\Omega_2)^3)$ (where domain Ω_2 satisfies $\overline{\Omega_2} \subset \Omega_1$), and this holds independently of a boundary condition. However, in the case of boundary condition (3), $\partial_t\boldsymbol{u}$ and ∇p (with all their space derivatives) are only known to belong to $L^\alpha(t_1+\zeta,\, t_2-\zeta;\, L^\infty(\Omega_2)^3)$ with $1<\alpha<2$. Considering boundary conditions (4), we can prove that $\partial_t\boldsymbol{u}$ and ∇p (with all their space derivatives) belong to $L^\infty(t_1+\zeta,\, t_2-\zeta;\, L^\infty(\Omega_2)^3)$. The proof, which can be found in [1], uses the fact that while in the case of the Dirichlet boundary condition (3), p satisfies the non–homogeneous Neumann boundary condition (11) with the not sufficiently controllable term $\nu\Delta\boldsymbol{u}\cdot\boldsymbol{n}$, the same problem can be simplified in the case of boundary conditions (4) because $\nu\Delta\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$. □

II. (A continuous family of solutions of the Euler and Navier–Stokes equations) Paper [2] shows that a locally in time continuous family of strong solutions of the Euler or Navier–Stokes equations in a bounded domain can be constructed, using slightly modified conditions (4) for solutions of the Navier–Stokes equation. This result represents a contribution to solution of one of the most important questions of mathematical fluid mechanics, i.e. the relation between solutions of the Euler and the Navier–Stokes equations. The same results are also known in the whole space $\mathbb{R}^3$ or in a bounded domain with space–periodic boundary conditions (see P. Constantin and C. Foias [3]), however a modification with the Dirichlet boundary condition (possibly also non–homogeneous) for solutions of the Navier–Stokes equation represents an open problem. □

The next theorem confirms that considering boundary conditions (4), we can prove a similar result to a classical theorem of K. K. Kiselev and O. A. Ladyzhenskaya (see [8]) which concerns the Dirichlet boundary condition (3). Moreover, a deeper analysis shows that $j = 2$ is the limit case and the theorem cannot be generalized for an arbitrary $j \in \mathbb{N}$.

Theorem 4 (Local in time existence of a strong solution) *Let* $j = 1$ *or* 2 *and* $\boldsymbol{u}_0 \in D^j$. *Then there exists* $T_j^* > 0$ *and a strong solution* $\boldsymbol{u}$ *of the problem (1), (2), (4) on the time interval* $(0, T_j^*)$ *such that* $\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0$, $\boldsymbol{u} \in C(0, T_j^*;\, D^j)$, $A^{j+1}\boldsymbol{u} \in L^2(0,T_j^*;\, L^2_\sigma(\Omega)^3)$ *and* $\partial_t A^{j-1}\boldsymbol{u} \in L^2(0,T_j^*;\, L^2_\sigma(\Omega)^3)$.

Principle of the proof: The n–th approximation $\boldsymbol{u}_n$ of solution $\boldsymbol{u}$ can be constructed as a linear combination of the eigenfunctions $\boldsymbol{e}^i$ of operator A for $i = \pm1, \dots, \pm n$ with coefficients a_i being functions of t so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \boldsymbol{u}_n\cdot\boldsymbol{e}^i\,\mathrm{d}\boldsymbol{x} + \int_\Omega (A\boldsymbol{u}_n\times\boldsymbol{u}_n)\cdot\boldsymbol{e}^i\,\mathrm{d}\boldsymbol{x} + \nu\int_\Omega A\boldsymbol{u}_n\cdot A\boldsymbol{e}^i\,\mathrm{d}\boldsymbol{x} \;=\; 0 \tag{18}$$

for all $i = \pm1, \dots, \pm n$ and $\boldsymbol{u}_n(\,.\,,0) = \Pi_n\boldsymbol{u}_0$. Multiplying the i–th equation in (18) by $\lambda_i^2\, a_i$ and summing over $i = \pm1, \dots, \pm n$, we obtain an inequality which provides a local in time estimate of $\|A\boldsymbol{u}_n\|_2$.

If we multiply the i–th equation in (18) by $\lambda_i^4\, a_i$ and sum over $i = \pm1, \dots, \pm n$, we obtain the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{1}{2}\int_\Omega |A^2\boldsymbol{u}_n|^2\,\mathrm{d}\boldsymbol{x} + \int_\Omega (A\boldsymbol{u}_n\times\boldsymbol{u}_n)\cdot A^4\boldsymbol{u}_n\,\mathrm{d}\boldsymbol{x} + \nu\int_\Omega |A^3\boldsymbol{u}_n|^2\,\mathrm{d}\boldsymbol{x} \;=\; 0. \tag{19}$$

It is important that we can integrate by parts in the second integral and rewrite it as a sum of the integral of $(A\boldsymbol{v}^n \times \boldsymbol{v}^n) \cdot (\boldsymbol{n} \times A^3\boldsymbol{v}^n)$ on $\partial\Omega$ and the integral of $\mathbf{curl}(A\boldsymbol{v}^n \times \boldsymbol{v}^n) \cdot A^3\boldsymbol{v}^n$ on Ω. The surface integral equals zero. (Indeed, $A\boldsymbol{v}^n$ and $\boldsymbol{v}^n$ are tangent on $\partial\Omega$, hence $A\boldsymbol{v}^n \times \boldsymbol{v}^n$ is normal, while $\boldsymbol{n} \times A^3\boldsymbol{v}^n$ is tangent.) The integral on Ω can be estimated by the sum of $\|A^2\boldsymbol{u}_n\|_2^2$ and $\|A^2\boldsymbol{u}_n\|_2^3$, multiplied by appropriate constants. This leads to estimates of $\|A^2\boldsymbol{u}_n\|_2$, respectively $\|A^3\boldsymbol{u}_n\|_2$, in $L^\infty(0, T_2^*)$, respectively in $L^2(0, T_2^*)$ (for some $T_2^* > 0$), which, for $n \to +\infty$, verify the statement of the theorem. □

Theorem 4 enables to re–derive the so called theorem on structure of a weak solution, which is well known in the case of the Dirichlet boundary condition (3), for a solution satisfying the generalized impermeability boundary conditions (4). The core of the theorem says that a solution $\boldsymbol{u}$ to problem (WP) that satisfies the strong energy inequality (16) is "smooth" on a union of open non–overlapping intervals in $(0, T)$ whose complement to $(0, T)$ has the 1–dimensional Hausdorff measure equal to zero. The rate of smoothness on each of the open intervals is given by Theorem 4. The theorem on structure will be applied in a forthcoming paper [10] which shows that one can use fine properties of a solution of problem (WP) (i.e. the weak problem with boundary conditions (4)) on and in the neighborhood of the boundary and extend "up to the boundary" some results on the interior regularity of $\boldsymbol{u}$. This concerns namely the result on regularity in dependence on the eigenvalues of the rate of deformation tensor, whose validity "up to the boundary" with the Dirichlet boundary condition (3) remains open.

7 Conclusion

The results explained in the previous sections show that the generalized impermeability boundary conditions (4) represent a mathematically acceptable alternative to (3). The question of their physical relevance on boundaries of various smoothness will be clearer after we shall have results of numerical computations and compare them with experimentally obtained data.

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