Proceedings of the 3rd IASME/WSEAS Int. Conf. on FLUID DYNAMICS & AERODYNAMICS, Corfu, Greece, August 20-22, 2005 (pp12-17)

Anisotropically Weighted Estimates of Weak Solutions to the Stationary Rotating Oseen Equations

STANISLAV KRAČMAR
Czech Technical University
Faculty of Mechanical Engineering
Department of Technical Mathematics
Karlovo nám. 13, 121 13 Praha 2
CZECH REPUBLIC
kracmar@fsik.cvut.cz

ŠÁRKA NEČASOVÁ
Academy of Sciences of the CR
Mathematical Institute
Žitná 25
110 00 Praha 1
CZECH REPUBLIC
matus@math.cas.cz

PATRICK PENEL
Université du Sud, Toulon-Var
Department of Mathematics
B.P. 20132, 83957 Toulon – La Garde
FRANCE
penel@univ-tln.fr

Abstract: - We study the rotating Oseen problem in anisotropically weighted Sobolev Spaces. We prove the existence and uniqueness of the solution.

Key-Words: - Anisotropically weighted Sobolev spaces, rotating Oseen problem.

1 Introduction

In a three-dimensional exterior domain Ω in $\mathbb{R}^3$, the classical Oseen problem [9] describes the velocity vector $\mathbf{u}$ and the associated pressure p by a linearized version of the incompressible Navier-Stokes equations as a perturbation of $\mathbf{v}_\infty$ the velocity at infinity; $\mathbf{v}_\infty$ is generally assumed to be constant in a fixed direction, say the first axis, $\mathbf{v}_\infty = |\mathbf{v}_\infty|\,\mathbf{e}_1$. In the next we denote $|\mathbf{v}_\infty|$ by k, and we will write the Oseen operator $k\,\partial_1\mathbf{v}$. On the other hand it is known that for various flows past a rotating obstacle, the Oseen operator appears in the form $(\mathbf{a}\cdot\nabla)\,\mathbf{v}$ with some concrete nonconstant coefficient functions, e.g. $\mathbf{a} = \omega\times\mathbf{x}$, where ω is an angular velocity. So, we investigate the following problem, so-called stationary rotating Oseen model,

$$-\nu\Delta\mathbf{u} + k\partial_1\mathbf{u} + (\omega\times\mathbf{x})\cdot\nabla\mathbf{u} \qquad (1)$$

$$-\omega\times\mathbf{u} + \nabla p = \mathbf{f} \ \text{ in } \boldsymbol{\Omega}$$

$$\operatorname{div}\mathbf{u} = 0 \ \text{ in } \ \boldsymbol{\Omega} \qquad (2)$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega \qquad (3)$$

$$\mathbf{u} \to \mathbf{0} \quad \text{as } |\mathbf{x}| \to \infty \qquad (4)$$

where ν and k are some positive constants, $\omega = (\lambda,\,0,\,0)$ is a constant vector. The vector function $\mathbf{u} = \mathbf{u}\,(\mathbf{x})$ describes an infinite incompressible fluid which is at rest at infinity, and the forcing term $\mathbf{f} = \mathbf{f}(\mathbf{x})$ is given.

Let us begin with some comment and relevant process of analysis of the problem (1)– (4). The governing fluid motion is essentially linear, but we are concerned with an exterior domain $\boldsymbol{\Omega}$, and the convective operators, $k\,\partial_1$ and $(\omega\times\mathbf{x})\cdot\nabla$, cannot be treated as perturbations of lower order of the Laplacian, this is well known.

A common approach to study the asymptotic properties of the solutions to the Dirichlet problem of the classical steady Oseen flow is to use convolutions with Oseen fundamental tensor and its first and second gradients for the velocity (or with the fundamental solution of Laplace equation for the pressure): L^q estimates in anisotropically weighted Sobolev spaces can be derived, see [6]. The fundamental solution to rotating Oseen problem in the time dependent case is known, see [10], but, unfortunately, the respective stationary kernel is not seem to be of Calderon-Zygmund type. The Littlewood-Paley theory offer another approach for an L^q-analysis: Thus, L^q estimates in nonweighted spaces were derived for the rotating Stokes problem by T. Hishida [4], and for the rotating Oseen problem in $\mathbb{R}^3$ by R. Farwig [2]. Looking for estimates in anisotropically weighted spaces, this approach generates increased technical difficulties. So, let us prefer a variational approach.

The same variational viewpoint has been already applied in [7] by S. Kracmar and P. Penel to solve the following generic scalar model of equation (1) with a given non-constant vector function a,

$$-\nu\,\Delta u + k\,\partial_1 u + a\cdot\nabla u = f \quad \text{in } \Omega \qquad (5)$$

together with boundary conditions $u=0$ on $\partial\Omega$ and $u\to 0$ as $|x|\to\infty$.

Introducing the chosen weight functions, to reflect the decay properties near the infinity, yields

$$w(x) = \eta_\beta^\alpha(\mathbf{x}) = \eta_\beta^\alpha(\mathbf{x};\delta,\varepsilon) = (1+\delta r)^\alpha\,(1+\varepsilon s)^\beta,$$

for $\mathbf{x}=[x_1,\,x_2,\,x_3]\in\mathbb{R}^3$, $\varepsilon,\,\delta>0$, $\alpha,\,\beta\in\mathbb{R}$,

$$r = |\mathbf{x}| = (x_1^2+x_2^2+x_3^2)^{1/2}, \quad s = s(\mathbf{x}) = r - x_1,$$

where δ and ε are useful to rescale separately the isotropic and anisotropic parts.

Discussing the range of the exponents α and β, the corresponding weighted spaces $L^p\left(\mathbb{R}^3;\,w\right)$ give the appropriate framework to test the solutions of both problems (5) and (1)–(4). Let us recall that η_β^α belongs to the Muckenhoupt class A_2 of weights in $\mathbb{R}^3$ if $-1<\beta<1$ and $-3<\alpha+\beta<3$.

In this paper we extend the results of [7] to the stationary rotating Oseen model (1), (2) and (4) on the whole space $\mathbb{R}^3$; we are concerned with $p=2$.

Our main result is

Theorem 1 *(Existence and uniqueness)*
Let $0<\beta\le 1,\ 0\le\alpha<y_1\cdot\beta,\ \ \mathbf{f}\in\mathbf{L}^2_{\alpha+1,\beta}$; y_1 *will be specified in Lemma 6.*

Then there exists a unique weak solution $\{\mathbf{u},\,p\}$ *of the problem (1), (2), (4) in the whole space* $\mathbb{R}^3$, *such that* $\mathbf{u}\in\mathbf{V}_{\alpha,\beta}$, $p\in L^2_{\alpha,\beta-1}$, $\nabla p\in\mathbf{L}^2_{\alpha+1,\beta}$ *and*

$$\|\mathbf{u}\|^2_{2,\alpha-1,\beta} + \|\nabla\mathbf{u}\|^2_{2,\alpha,\beta} + \|p\|^2_{2,\alpha,\beta-1} + \|\nabla p\|^2_{2,\alpha+1,\beta} \le C\,\|\mathbf{f}\|^2_{2,\alpha+1,\beta}.$$

2 Notations and Function Spaces

Let us outline our notations:

$$B_R = \left\{x\in\mathbb{R}^3;\ |x|\le R\right\},$$

$$B^R = \left\{x\in\mathbb{R}^3,\ |x|\ge R\right\}.$$

Let, for $1\le q<\infty$,

$$D^{m,q}(\Omega) = \left\{u\in L^1_{loc}(\Omega):\ D^l u\in L^q(\Omega)\ |l|\le m\right\}$$

with $|u|_{m,q} = \left(\sum_{|l|=m}\int_\Omega \left|D^l u\right|^q\right)^{1/q}$ as a seminorm. It is known that $D^{m,q}(\Omega)$ is a Banach space (and if $q=2$ a Hilbert space), provided we identify two functions $u_1,\,u_2$ whenever $|u_1-u_2|_{m,q}=0$, i. e. $u_1,\,u_2$ differs (at most) on a polynomial function of the degree $m-1$.

Let $\left(L^2\left(\mathbb{R}^3;\,w\right)\right)^3$ be the set of measurable vector functions $\mathbf{f}$ on $\mathbb{R}^3$ such that

$$\|\mathbf{f}\|_{2,\mathbb{R}^3;\,w} = \left(\int_{\mathbb{R}^3}|\mathbf{f}|^2\,w\,d\mathbf{x}\right)^{1/2} < \infty.$$

We will use $\mathbf{L}^2_{\alpha,\beta}$ instead of $\left(L^2\left(\mathbb{R}^3;\,\eta_\beta^\alpha\right)\right)^3$ and $\|\cdot\|_{2,\alpha,\beta}$ instead of $\|\cdot\|_{2,\mathbb{R}^3;\eta_\beta^\alpha}$. Because $\left(\eta_\beta^\alpha\right)^{-1}$ is locally integrable, then, by Hölder's inequality, it follows that $\mathbf{L}^2_{\alpha,\beta}\subset\left(L^1_{loc}\left(\mathbb{R}^3\right)\right)^3$. It thus makes sense to talk about weak derivatives of functions in $\mathbf{L}^2_{\alpha,\beta}$. Let us define the weighted Sobolev space $\mathbf{H}^1\left(\mathbb{R}^3;\,\eta_{\beta_0}^{\alpha_0},\eta_{\beta_1}^{\alpha_1}\right)$ as the set of functions $\mathbf{u}\in\mathbf{L}^2_{\alpha_0,\beta_0}$ with the weak derivatives $\partial_i\mathbf{u}\in\mathbf{L}^2_{\alpha_1,\beta_1}$. The norm of $\mathbf{u}\in\mathbf{H}^1\left(\mathbb{R}^3;\,\eta_{\beta_0}^{\alpha_0},\eta_{\beta_1}^{\alpha_1}\right)$ is given by

$$\|\mathbf{u}\|^2_{\mathbf{H}^1\left(\mathbb{R}^3;\,\eta_{\beta_0}^{\alpha_0},\eta_{\beta_1}^{\alpha_1}\right)} = \|\mathbf{u}\|^2_{2,\alpha_0,\beta_0} + \|\nabla\mathbf{u}\|^2_{2,\alpha_1,\beta_1}.$$

As usual, $\mathring{\mathbf{H}}^1\left(\mathbb{R}^3;\,\eta_{\beta_0}^{\alpha_0},\eta_{\beta_1}^{\alpha_1}\right)$ will be the closure of $\left(C_0^\infty\right)^3$ in $\mathbf{H}^1\left(\mathbb{R}^3;\,\eta_{\beta_0}^{\alpha_0},\eta_{\beta_1}^{\alpha_1}\right)$. For simplicity, we shall use the following abbreviations:

$\mathring{\mathbf{H}}^1_{\alpha,\beta}$ instead of $\mathring{\mathbf{H}}^1\left(\mathbb{R}^3;\,\eta_{\beta-1}^{\alpha-1},\eta_\beta^\alpha\right)$

$\mathbf{V}_{\alpha,\beta}$ instead of $\mathring{\mathbf{H}}^1\left(\mathbb{R}^3;\,\eta_{\beta}^{\alpha-1},\eta_\beta^\alpha\right)$

In fact we shall only use these last two Hilbert spaces for $\alpha\ge 0,\ \beta>0,\ \alpha+\beta<3$, and $\mathring{\mathbf{H}}^1$ without indices when it is the usual Sobolev space on bounded domain, e.g. B_R.

3 Some Auxiliary Results

The weighted estimates of the solution to the stationary classical Oseen problem were firstly obtained by R. Finn [3] in 1959, and then improved by R. Farwig [2] in 1992. See [7] for other comments and references.

The case of equation (1) with the rotation effect is worth thinking over: Let us assume for a moment that pressure p is known. In solving the problem (1) (4) with respect to $\mathbf{u}$ by means of a pure variational approach, we shall deal with the following equation:

$$\nu \int_{\mathbb{R}^3} |\nabla \mathbf{u}|^2 \, w \, d\mathbf{x} + \nu \int_{\mathbb{R}^3} (\nabla w \cdot \nabla) \, \mathbf{u} \cdot \mathbf{u} \, d\mathbf{x}$$
$$-\frac{1}{2} \int_{\mathbb{R}^3} |\mathbf{u}|^2 \left[k \partial_1 w + \operatorname{div} \left(w \left[\omega \times \mathbf{x} \right] \right) \right] d\mathbf{x} \qquad (6)$$
$$= \int_{\mathbb{R}^3} \mathbf{f} \cdot \mathbf{u} \, w \, d\mathbf{x} - \int_{\mathbb{R}^3} \nabla p \cdot \mathbf{u} \, w \, d\mathbf{x}$$

as we get integrating formally the product of (1) and $\mathbf{u}\, w$ with w an appropriate weight function. It is not difficult to observe that for $w = \eta_\beta^\alpha$ we have $\operatorname{div}(w\,[\omega \times \mathbf{x}]) = 0$. The left hand side in (6) can be estimated from below by

$$\frac{\nu}{2} \int_{\mathbb{R}^3} |\nabla \mathbf{u}|^2 \, w \, d\mathbf{x} \qquad (7)$$
$$+\frac{1}{2} \int_{\mathbb{R}^3} |\mathbf{u}|^2 \left(-\nu \frac{|\nabla w|^2}{w} - k \partial_1 w \right) d\mathbf{x}$$

and it can be proved the existence of $s_0 > 0$ such that $-\nu \frac{|\nabla w|^2}{w} - k\partial_1 w \geq 0$ for $w = \eta_\beta^\alpha$ and $s \geq s_0$, (see [7, Appendix A]). Moreover, because this term is known explicitly, we have the possibility to evaluate it from below by a "small" negative quantity in the form $-C(\alpha, \beta, \delta, \epsilon)\, \eta_{\beta-1}^{\alpha-1}$ without any constraint in $s(\cdot)$, see Lemma 2 hereafter.

Another useful preliminary remark is that of a generalized Friedrichs-Poincaré type inequality in $\mathring{\mathbf{H}}^1_{\alpha,\beta}$. This leads to the Lemma 3 which is the first main technical result of this paper. The obtained inequality allows us to compensate by the viscous Dirichlet integral the "small" negative contribution coming from the second integral of (7). Therefore the existence of a weak solution to problem (1), (2), and (4) in $\mathbf{V}_{\alpha,\beta}$ can be proven essentially by the Lax-Milgram theorem.

Let us define a function $F_{\alpha,\beta}(s, r;\, \nu)$ by the relation:

$$F_{\alpha,\beta}\,(s, r;\, \nu) \cdot \eta_{\beta-1}^{\alpha-1} \equiv -\nu \frac{\left| \nabla \eta_\beta^\alpha \right|^2}{\eta_\beta^\alpha} - k\, \partial_1\, \eta_\beta^\alpha \qquad (8)$$

We now summarize the main auxiliary results:

Lemma 2 *(From [7]) Let* $0 \leq \alpha < \beta$, $\kappa > 1$, $0 < \varepsilon \leq \frac{1}{2\kappa} \cdot \frac{k}{\nu} \cdot \frac{\beta - \alpha}{\beta^2}$ *and* $\delta, \nu, k > 0$. *Then*

$$F_{\alpha,\beta}\,(s, r; \nu) - \left(1 - \frac{1}{\kappa}\right) \cdot k \cdot \delta \cdot \varepsilon \cdot (\beta - \alpha) \cdot s$$
$$\geq -\alpha\, \delta\, k \left(1 + \frac{\nu}{k}\, \alpha\, \delta\right)$$

for all $r > 0$ *and* $s \in [0, 2r]$.

Lemma 3 *(Friedrichs-Poincaré type inequality) Let* $\alpha \geq 0$, $\beta > 0$, $\alpha + \beta < 3$, $\kappa > 1$. *Let* δ *and* ε *be arbitrary positive constants, such that* $(\beta - \alpha)\,(2\varepsilon - \delta) \geq 0$. *Then for all* $\mathbf{u} \in \mathring{\mathbf{H}}^1_{\alpha,\beta}$

$$\int_{\mathbb{R}^3} \mathbf{u}^2\, \eta_{\beta-1}^{\alpha-1}\, d\mathbf{x}$$
$$\leq \frac{(\alpha\,\delta + 2\,\beta\,\varepsilon)^2}{(\beta\,\beta^*\,\delta\,\varepsilon)^2} \int_{B_{R_0}} |\nabla \mathbf{u}|^2\ \eta_\beta^\alpha\, d\mathbf{x}$$
$$+ \frac{2\,\kappa\,(\alpha + \beta)^2}{(\beta\,\beta^*)^2\ \delta\,\varepsilon} \int_{B^{R_0}} |\nabla \mathbf{u}|^2\ \eta_\beta^\alpha\, d\mathbf{x},$$

where $R_0 \geq \left| \frac{1}{\delta} - \frac{1}{2\varepsilon} \right| \frac{1}{(\kappa - 1)}$. *Moreover, if* $\delta = 2\varepsilon$ *then*

$$\|\mathbf{u}\|_{2,\alpha-1,\beta-1} \leq \left(\frac{\alpha + \beta}{\beta\,\beta^*\varepsilon} \right) \|\nabla \mathbf{u}\|_{2,\alpha,\beta}\,. \qquad (9)$$

For the proof see [8]. The same inequality holds in $\mathring{\mathbf{H}}^1_{\alpha,\beta}(\Omega)$ for an exterior domain Ω. Its scalar variant was given and proved in [7].

To prove uniqueness we will need also a classical result, the following auxiliary result about weakly harmonic functions in $D^{1,q}(\mathbb{R}^n)$:

Lemma 4 *Let* $n \geq 2$ *and let* $v \in D^{1,q}(\mathbb{R}^n)$ *with* $1 \leq q < \infty$ *such that*

$$\int_{\mathbb{R}^n} v\ \triangle\, \phi\, dx = 0 \qquad \text{for all } \phi \in C_0^\infty(\mathbb{R}^n)\,.$$

Then there is a constant C *such that* $v(x) = C$ *a.e. in* $\mathbb{R}^n$.

4 Existence of Weak Solution in $\mathbb{R}^3$

For technical reasons, we now assume $0 < \beta \leq 1$, $0 \leq \alpha < y_1 \cdot \beta$, where the parameter y_1 will be the same as in Lemma 6. Let $\mathbf{f} \in \mathbf{L}^2_{\alpha+1,\beta}$, we want to sketch the proof of existence in Theorem 1.

Step **1.** If there exist distributions $\mathbf{u}, p$ satisfying (1), (2) then pressure p satisfies the equation

$$\triangle p = \operatorname{div} \mathbf{f} \qquad (10)$$

because

$$\operatorname{div}\left((\omega \times \mathbf{x}) \cdot \nabla \mathbf{u} - \omega \times \mathbf{u}\right) = (\omega \times \mathbf{x}) \cdot \nabla \operatorname{div} \mathbf{u} = 0$$

(and of course $\operatorname{div}(\Delta \mathbf{u} + k\, \partial_1 \mathbf{u}) = 0$). Let $\mathcal{E}$ be the fundamental solution of the Laplace equation, i.e.

$$\mathcal{E} = -\frac{1}{4\pi} \frac{1}{r}.$$

Assuming firstly $\mathbf{f} \in (C_0^\infty)^3$ we have $p = \mathcal{E} \star \operatorname{div}\mathbf{f}$ and $\nabla p = \nabla\mathcal{E} \star \operatorname{div}\mathbf{f}$, and so $p = \nabla\mathcal{E} \star f$ and $\nabla p = \nabla^2\mathcal{E} \star \mathbf{f}$.

It is well known that both formulas can be extended for $\mathbf{f} \in \mathbf{L}^2_{\alpha+1,\beta}$ with $0 \le \alpha < \beta < 1$, the last convolution $\nabla p = \nabla^2\mathcal{E} \star \mathbf{f}$ due to the fact that $\nabla^2\mathcal{E}$ is a singular kernel of the Calderon-Zygmund type and that $\eta_\beta^{\alpha+1}$ belongs to the Muckenhoupt class of weights A_2: see [1, Thm. 3.2, Thm 5.5] and [6, Thm. 4.4,Thm 5.4], where the theorems are formulated for the pressure part $\mathcal{P}$ of the fundamental solution of the classical Oseen problem, so $\mathcal{P} = \nabla\mathcal{E}$ and $\nabla\mathcal{P} = \nabla^2\mathcal{E}$.

For $\mathbf{f} \in \mathbf{L}^2_{\alpha+1,\beta}$ we get $p \in L^2_{\alpha,\beta-1}$ and $\nabla p \in \mathbf{L}^2_{\alpha+1,\beta}$ and there are constants C_1, $C_2 > 0$ such that the following estimates are satisfied:

$$\|p\|^2_{2,\alpha,\beta-1} \le C_1 \|\mathbf{f}\|^2_{2,\alpha+1,\beta}$$

$$\|\nabla p\|^2_{2,\alpha+1,\beta} \le C_2 \|\mathbf{f}\|^2_{2,\alpha+1,\beta}$$

Step **2.** We prove the existence of a weak solution $\mathbf{u}_R \in \mathring{\mathbf{H}}^1(B_R)$ to the following problem on B_R:

$$-\nu\cdot\Delta\mathbf{u} + k\cdot\partial_1\mathbf{u} + (\omega\times\mathbf{x})\cdot\nabla\mathbf{u}$$

$$-\omega\times\mathbf{u} = \mathbf{f} - \nabla p \quad \text{in } B_R \tag{11}$$

$$u = 0 \quad \text{on } \partial B_R \tag{12}$$

the right hand side $\mathbf{f} - \nabla p$ being known in $\mathbf{L}^2_{\alpha+1,\beta}$.

Let us introduce a continuous bilinear form $Q_1(\cdot,\cdot)$ on $\mathring{\mathbf{H}}^1(B_R) \times \mathring{\mathbf{H}}^1(B_R)$ with $\beta_0 \in (0,1]$:

$$\begin{aligned} Q_1(\mathbf{u},\mathbf{v}) \;=\; & \nu \int_{B_R} \nabla\mathbf{u}\cdot\nabla\left(\mathbf{v}\cdot\eta^0_{\beta_0}\right)\cdot d\mathbf{x} \\ & + k\int_{B_R} \partial_1\mathbf{u}\cdot\left(\mathbf{v}\cdot\eta^0_{\beta_0}\right)\cdot d\mathbf{x} \\ & + \int_{B_R} (\omega\times\mathbf{x})\cdot\nabla\mathbf{u}\,\left(\mathbf{v}\,\eta^0_{\beta_0}\right)\cdot d\mathbf{x} \end{aligned}$$

Using notation (8), we have:

$$\begin{aligned} Q_1(\mathbf{v},\mathbf{v}) \;\ge\; & \frac{\nu}{2}\int_{B_R} |\nabla\mathbf{v}|^2\,\eta^0_{\beta_0}\,d\mathbf{x} \\ & + \frac{1}{2}\int_{B_R} \mathbf{v}^2\,F_{0,\beta_0}(s,r;\,\nu)\cdot\eta^{-1}_{\beta_0-1}d\mathbf{x} \end{aligned}$$

From Lemma 2 with $\alpha = 0$ and the Lax-Milgram theorem we get:

Lemma 5 *Let* $0 < \beta_0 \le 1$*. Then, for all* $\mathbf{f} \in \mathbf{L}^2_{1,\beta_0}(B_R)$, $\varepsilon_0 < \frac{1}{2}\cdot\frac{k}{\nu}\cdot\frac{1}{\beta_0}$, $\eta^\alpha_{\beta_0} \equiv \eta^{\alpha,\varepsilon_0}_{\beta_0,\varepsilon_0}$*. There exists* $\mathbf{u}_R \in \mathring{\mathbf{H}}^1(B_R)$*, the unique solution of*

$$Q_1(\mathbf{u}_R,\mathbf{v}) = \int_{B_R} (\mathbf{f} - \nabla p)\cdot\mathbf{v}\,\eta^0_{\beta_0}\,d\mathbf{x} \tag{13}$$

for all $\mathbf{v} \in \mathring{\mathbf{H}}^1(B_R)$ *.*

Step **3.** Our next aim is to get uniform estimates of $\mathbf{u}_R$ in $\mathbf{V}_{\alpha,\beta}$ as $R \to +\infty$. Let y_1 be the unique real solution of the algebraic equation $4y^3+8y^2+5y-1 = 0$. It is easy to verify that $y_1 \in (0,1)$. The control of α/β is necessary for the compatibility of all conditions on α, β, δ, ε, κ, see [7].

Lemma 6 *Let* $0 < \beta \le 1$, $0 \le \alpha < y_1\cdot\beta$, $\mathbf{f} - \nabla p \in \mathbf{L}^2_{\alpha+1,\beta}$*. Then, as* $R \to +\infty$*, the weak solutions* $\mathbf{u}_R$ *of (13) given by Lemma 5 are uniformly bounded in* $\mathbf{V}_{\alpha,\beta}$*. There is a constant* $C > 0$*, which does not depend on* R *such that*

$$\int_{\mathbb{R}^3} \tilde{\mathbf{u}}_R^2\cdot\eta_\beta^{\alpha-1}\cdot d\mathbf{x} + \int_{\mathbb{R}^3} |\nabla\tilde{\mathbf{u}}_R|^2\cdot\eta^\alpha_\beta\cdot d\mathbf{x} \tag{14}$$

$$\le C\int_{\mathbb{R}^3}\left(|\mathbf{f}|^2 + |\nabla p|^2\right)\cdot\eta_\beta^{\alpha+1}\cdot d\mathbf{x}$$

for all R *greater than some* $R_0 > 0$, $\tilde{\mathbf{u}}_R$ *being extension by zero of* $\mathbf{u}_R$ *on* $\mathbb{R}^3 \setminus B_R$*.*

For the proof see [8]. The same ideas were used in the proof of [7, Lemma 3.4]. First, we need an uniform estimate of the expression

$$\int_{B_{R_1}} \tilde{\mathbf{u}}_R^2\cdot\eta_\beta^{\alpha-1}\cdot d\mathbf{x} + \int_{B_{R_1}} |\nabla\tilde{\mathbf{u}}_R|^2\cdot\eta^\alpha_\beta\cdot d\mathbf{x}$$

for some sufficiently large and fixed $R_1 > 0$. Secondly, using the Friedrichs-Poincaré type inequality from Lemma 3, we get (14).

Step **4.** Let $\{R_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers converging to $+\infty$. Let $\mathbf{u}_{R_n}$ be the weak solution of (11), (12) on B_{R_n}. Extending $\mathbf{u}_{R_n}$ by zero on $\mathbb{R}^3 \setminus B_{R_n}$ to a function $\tilde{\mathbf{u}}_\mathbf{n} \in \mathbf{V}_{\alpha,\beta}$ we get a bounded sequence $\{\tilde{\mathbf{u}}_\mathbf{n}\}_n$ in $\mathbf{V}_{\alpha,\beta}$. Thus, there is a subsequence $\{\tilde{\mathbf{u}}_{\mathbf{n_k}}\}_k$ with a weak limit $\mathbf{u}$ in $\mathbf{V}_{\alpha,\beta}$. Obviously, $\{\mathbf{u},\,p\}$ is a weak solution of (1) and we have

$$\begin{aligned} & \|\mathbf{u}\|_{2,\alpha-1,\beta}^2 + \|\nabla \mathbf{u}\|_{2,\alpha,\beta}^2 \\ \leq\ & \liminf_{k\in\mathbb{N}} \left(\int_{\mathbb{R}^3} \tilde{\mathbf{u}}_{n_k}^2\, \eta_\beta^{\alpha-1}\, d\mathbf{x} + \int_{\mathbb{R}^3} |\nabla \tilde{\mathbf{u}}_{n_k}|^2\, \eta_\beta^{\alpha}\, d\mathbf{x} \right) \\ \leq\ & C_3 \int_{\mathbb{R}^3} \left(|\mathbf{f}|^2 + |\nabla p|^2 \right) \cdot \eta_\beta^{\alpha+1} \cdot d\mathbf{x} \\ \leq\ & C_4 \int_{\mathbb{R}^3} |\mathbf{f}|^2\, \eta_\beta^{\alpha+1}\, d\mathbf{x}, \end{aligned}$$

hence the estimate from Theorem 1 is satisfied. To complete the proof, it remains to check that $\mathbf{u}$ solves also the equation

$$\operatorname{div} \mathbf{u} = 0 \text{ a.e. in } \mathbb{R}^3,$$

which we do in the last section because a similar idea of proof states the uniqueness of $\mathbf{u}$.

5 Solenoidality and Uniqueness of the Weak Solution

Let us mention that from the properties of the gradient of $\mathbf{u}$ follows $\operatorname{div} \mathbf{u} \in L^2_{\alpha,\beta}$, and that $\mathbf{u} \in \mathbf{H}^2_{loc}$ because $\mathbf{f} - \nabla p \in \mathbf{L}^2_{\alpha+1,\beta}$. So, applying weakly the operator div on equation (1), we get

$$\begin{aligned} & -\nu\, \Delta\, (\operatorname{div} \mathbf{u}) + k\partial_1\, (\operatorname{div} \mathbf{u}) \\ & \quad + (\omega \times \mathbf{x}) \cdot \nabla\, (\operatorname{div} \mathbf{u}) \\ & = \operatorname{div} \mathbf{f} - \triangle p = 0. \end{aligned} \tag{15}$$

Lemma 7 *Let* $\mathbf{u}$ *be the weak limit obtained in* $\mathbf{V}_{\alpha,\beta}$ *and* p *obtained in* $L^2_{\alpha,\beta-1}$. *Then:*
(i) $\operatorname{div} \mathbf{u} \in V_{0,\beta}$ *and the norm of* $\operatorname{div} \mathbf{u}$ *in the space* $V_{0,\beta}$ *is zero;*
(ii) if $\mathbf{f} = 0$, $p = 0$ *and the norm of* $\mathbf{u}$ *in* $V_{0,\beta}$ *is zero.*

To prove this lemma, at first we observe that with $\mathbf{f} = 0$ we necessarily have $\triangle p = 0$ in $\mathbb{R}^3$, then using Lemma 4, we get that $p = 0$. Therefore the same (scalar or vectorial) equation

$$-\nu\, \Delta\gamma + k\, \partial_1\gamma + (\omega \times \mathbf{x}) \cdot \nabla\gamma = 0 \tag{16}$$

can be used to describe in $\mathbb{R}^3$ either $\gamma = \operatorname{div} \mathbf{u}$ or $\gamma = \mathbf{u}$.

Let us define a convenient cut-off function Φ_R : If $\Phi = \Phi(z) \in C_0^\infty(\langle 0, +\infty))$ is a non-increasing function such that $\Phi(z) \equiv 1$ for $z < \frac{1}{2}$, $\Phi(z) \equiv 0$ for $z > 1,$ and $|\Phi'| \leq 3$, we take $\Phi_R(x) \equiv \Phi\left(\frac{|x|}{R}\right)$, then we have $|\nabla \Phi_R(x)| \leq 3 \cdot \frac{1}{R}$ and $|\partial_1 \Phi_R| \leq 3 \cdot \frac{1}{R}$ for $x \in \Omega$, $\frac{R}{2} \leq |x| \leq R$.

Let $\{R_j\}_j$ be an increasing sequence of radii in $\mathbb{R}$ with the limit $+\infty,$ and let us denote $\gamma_j \equiv \gamma \cdot \Phi_{R_j}$ So, $\{\gamma_j\}_j$ is a sequence of functions either with limit $\gamma = \operatorname{div} \mathbf{u}$ in the space $L^2_{\alpha,\beta},$ or with limit $\gamma = \mathbf{u}$ in the space $\mathring{\mathbf{H}}^1$.

Using the test functions $\gamma_j \cdot \Phi_{R_j} \cdot (1 + \varepsilon s)^\beta \in \mathring{H}^1$ (i.e. $\operatorname{div} \mathbf{u} \cdot \Phi^2_{R_j} \cdot (1+\varepsilon s)^\beta$, or $\mathbf{u} \cdot \Phi^2_{R_j} \cdot (1+\varepsilon s)^\beta \in \mathring{\mathbf{H}}^1$) in (16) we get:

$$\begin{aligned} & \nu \int_{\mathbb{R}^3} \nabla\gamma \cdot \nabla \left(\gamma \cdot \Phi^2_{R_j} \cdot \eta^0_\beta \right) \cdot d\mathbf{x} \\ & + k \int_{\mathbb{R}^3} \partial_1\gamma \cdot \gamma \cdot \Phi^2_{R_j} \cdot \eta^0_\beta \cdot d\mathbf{x} \\ & + \int_{\mathbb{R}^3} (\omega \times \mathbf{x}) \cdot \nabla\gamma \cdot \gamma \cdot \Phi^2_{R_j} \cdot \eta^0_\beta \cdot d\mathbf{x} = 0. \end{aligned}$$

Integrating by parts, we get after some rearrangements

$$\begin{aligned} & \left(1 - \frac{1}{\kappa}\right) \frac{\nu}{2} \int_{\mathbb{R}^3} |\nabla\gamma_j|^2 \cdot \eta^0_\beta \cdot d\mathbf{x} \\ & + \frac{1}{2} \left(1 - \frac{1}{\kappa}\right) \cdot k \cdot \varepsilon_0^2 \cdot \beta_0 \cdot \int_{\mathbb{R}^3} \gamma_j^2 \cdot \eta^{-1}_{\beta-1} \cdot s \cdot d\mathbf{x} \\ \leq\ & C \int_{B^{R_j/2}_{R_j}} \gamma^2 \cdot \eta_\beta^{-1} \cdot dx, \end{aligned}$$

hence

$$\int_{\mathbb{R}^3} |\nabla\gamma|^2 \cdot \eta^0_\beta \cdot dx + \int_{\mathbb{R}^3} \gamma^2 \cdot \eta^{-1}_{\beta-1} \cdot s \cdot dx \leq 0,$$

and the solenoidality of $\mathbf{u}$ is proved. Replacing γ by $\mathbf{u}$ in the last inequality, we get $\mathbf{u} = \mathbf{0}$ and the uniqueness in $\mathbf{V}_{0,\beta} \supset \mathbf{V}_{\alpha,\beta}$.

6 Concluding Remarks: Extension to Exterior Domain

For simplicity we have yet limited our study to the whole space. The obtained results can be extended also for the exterior domain D, see [8]. First, let us mention that using the known

Lemma 8 *(Borchers and Sohr) Let* $g \in W_0^{k,p}(G)$. *Then there exists* $\mathbf{u_0} \in \mathbf{W}_0^{k+1,p}(G)$ *such that* $div\, \mathbf{u_0} = g$.

one can generalize Theorem 1 to the case when (2) is replaced by

$$\operatorname{div} \mathbf{u} = g \ \text{ in } \ \mathbb{R}^3. \tag{17}$$

We need only the case $k = 0$, $p = 2$ and G is a bounded domain.

For the extension of Theorem 1 to the case of an exterior domain we use the localization procedure, see [5]. By use of cut-off function Ψ we decompose the solution $\{\mathbf{u}, p\}$ of the problem (1)–(4) on the solution of the problem in $\mathbb{R}^3$ and the solution of some problem in a bounded domain:

$$\begin{aligned} \mathbf{u} &= \mathbf{U} + \mathbf{V} \quad \text{where } \mathbf{U} = (1-\Psi)\,\mathbf{u}, \quad \mathbf{V} = \Psi u, \\ p &= \sigma + \tau \quad \text{where } \sigma = (1-\Psi)\,p, \quad \tau = \Psi p, \end{aligned}$$

where $\Psi \in C_0^\infty$, $\operatorname{supp}\Psi \subset\subset B_{\rho_1}$ such that $\Psi \equiv 1$ on B_{ρ_0}, $\rho > \rho_1 > \rho_0 > 0$ so that $\mathbb{R}^3 \setminus D \subset B_{\rho_0}$. It can be shown that $\{\mathbf{U}, p\}$ and $\{\mathbf{V}, \tau\}$ are solutions of problems (with modified right-hand sides) (1), (17), (4) and the Stokes problem on a bounded domain, respectively.

To solve the Stokes problem on the bounded domain we use the following lemma, see [5]:

Lemma 9 *(*Cattabriga, Solonnikov, Kozono-Sohr*) Let* G *be a bounded domain with smooth boundary* ∂G *and let* $1 < q < \infty$. *Suppose that*

$$\mathbf{f} \in \mathbf{W}^{-1,q}(G), \qquad g \in L^q(G), \qquad \int_G g\, d\mathbf{x} = 0.$$

Then the problem

$$-\triangle \mathbf{V} + \nabla\tau = \mathbf{f} \quad \text{in } G \tag{18}$$

$$div\, \mathbf{V} = g \quad \text{in } G \tag{19}$$

$$\mathbf{u} = 0 \quad \text{on } \partial G \tag{20}$$

has a unique (up to additive constant for τ *) weak solution* $\{\mathbf{V}, \tau\} \in \mathbf{W}_0^{1,q}(G) \times L^q(G)$ *subject to the estimate*

$$\|\nabla \mathbf{V}\|_{q,G} + \|\tau - \overline{\tau}\|_{q,G} \le C\left(\|\mathbf{f}\|_{-1,q,G} + \|g\|_{q,G}\right),$$

where $\overline{\tau} = |G|^{-1}\int_G \tau\, d\mathbf{x}$.

Using now modified Theorem 1 for the problem (1), (17), (4) and Lemma 9 we get the result for exterior domains.

Acknowledgements: The authors gratefully acknowledge and appreciate the hospitality of the Mathematical Institute of the Academy of Sciences of the Czech Republic and of the University in Toulon. The research was supported by grant No. IAA100190505 from the Academy of Sciences of the Czech Republic, by grant No. 201050005 from the Grant Agency of the Czech Republic, by the project No. MSM 6840770010 of the Ministry of Education of the Czech Republic, and by the Academy of Sciences of the Czech Republic, Institutional Research Plan No. AV0Z10190503.

References:

[1] Farwig R.: The stationary exterior 3-D problem of Oseen and Navier-Stokes equations in anisotropically weighted Sobolev spaces. *Mathematische Zeitschrift* **211** (1992) 409-447.

[2] Farwig R.: An L^q−analysis of viscous fluid flow past a rotating obstacle. Preprint.

[3] Finn R.: Estimates at infinity for stationary solution of Navier–Stokes equations. *Bult. Math. de la Soc. Sci. Math. de la R.P.R.* Tome 3, **53** 4. (1959) 387–418.

[4] Hishida, T.: L^q estimates of weak solutions to the stationary Stokes equations around a rotating body. Preprint.

[5] Kozono H., Sohr H.: New apriory estimates for the Stokes equations in exterior domains. *Indiana Univ. Math. J.* **41** (1991) 1-27.

[6] Kračmar S., Novotný A., Pokorný M.: Estimates of Oseen kernels in weighted L^p spaces. *J. Math. Soc. Japan* 5**3**, No. 1 (2001) 59-111.

[7] Kračmar S., Penel P.: Variational properties of a generic model equation in exterior 3D domains. *Funkcialaj Ekvacioj*, **47**(2004) 499-523.

[8] Kračmar S., Nečasová Š., Penel P.: Anisotropic estimates of weak solutions to the stationary Oseen equations around a rotating body. Preprint.

[9] Oseen C.W.: Neuere Methoden und Ergebnisse in der Hydrodynamik. Leipzig, Akad. Verlagsgesellschaft M.B.H., 1927.

[10] Thomann, E.A., Guenther, R.B.:The fundamental solution of the linearized Navier-Stokes equations for spinning bodies in three spatial dimensions - time dependent case. Preprint, 2003.