Proceedings of the 3rd IASME/WSEAS Int. Conf. on HEAT TRANSFER, THERMAL ENGINEERING AND ENVIRONMENT, Corfu, Greece, August 20-22, 2005 (pp382-386)

Approximate analytical three-dimensional solution for periodical system with rectangular fin, Part 1

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Abstract: - In this paper the approximate analytical three dimensional solution for one element in periodical system with rectangular fins is obtained by the original method of conservative averaging.

Key-Words: - heat exchange, rectangular fin, steady state, three-dimensional, analytical solution, conservative averaging.

1 Introduction

Obtaining sufficient cooling for the components of devices is a difficult challenge in modern industry. It is related to refrigerators, radiators, engines and modern electronics, etc.

Usually its mathematical modeling is realized by one dimensional steady-state assumptions [1], [2], [8],[9]. In our previous papers [3] – [6] we have constructed two dimensional analytical approximate [3] – [5] and exact [6] solutions. In this paper we obtain approximate analytical three dimensional solution by the original method of conservative averaging and some its simplifications (special cases).

2 Mathematical Formulation of 3-D Problem

We will start with accurate three-dimensional formulation of steady-state problem for one element of periodical system with rectangular fin. This mathematical formulation is similar to those which are given in our papers [3]-[7] for 2-D case.

2.1 Description of Temperature Field in the Wall

We will use following dimensionless arguments and parameters [4]-[7]:

$$x = \frac{x'}{B+R}, y = \frac{y'}{B+R}, z = \frac{z'}{B+R}, \delta = \frac{\Delta}{B+R},$$

$$l = \frac{L}{B+R}, \; b = \frac{B}{B+R}, \; \beta_0^0 = \frac{h_0(B+R)}{k_0},$$

$$\beta_0 = \frac{h(B+R)}{k_0}, \; \beta = \frac{h(B+R)}{k}, \; w = \frac{W}{B+R},$$

where $k(k_0)$ - heat conductivity coefficient for the fin (wall), $h(h_0)$ - heat exchange coefficient for the fin (wall), 2B – width (thickness) of the fin, L – length of the fin , Δ - thickness of the wall, W – width (length) of the wall, 2R – distance between two fins (fin spacing).

The wall (base) is placed in the domain $\{x \in [0,\delta], y \in [0,1], z \in [0,w]\}$ and we describe the dimensionless temperature field $V_0(x,y,z)$ in the wall with the equation:

$$\frac{\partial^2 V_0}{\partial x^2} + \frac{\partial^2 V_0}{\partial y^2} + \frac{\partial^2 V_0}{\partial z^2} = 0. \quad (1)$$

We add needed boundary conditions as follow:

$$\frac{\partial V_0}{\partial x} + \beta_0^0 (1 - V_0) = 0, \; x = 0, \quad (2)$$

$$\frac{\partial V_0}{\partial x} + \beta_0 V_0 = 0, \; x = \delta, y \in (b,1), z \in (0,w), \quad (3)$$

$$\left.\frac{\partial V_0}{\partial y}\right|_{y=0} = \left.\frac{\partial V_0}{\partial y}\right|_{y=1} = 0, \quad (4)$$

$$\left.\frac{\partial V_0}{\partial z}\right|_{z=0} = 0, \quad \frac{\partial V_0}{\partial z} + \beta_0 V_0 = 0, \; z = w. \quad (5)$$

We assume the conjugations conditions on the surface between the wall and the fin as ideal thermal contact - there is no contact resistance:

$$V_0\big|_{x=\delta-0} = V\big|_{x=\delta+0}, \qquad (6)$$

$$\beta \frac{\partial V_0}{\partial x}\bigg|_{x=\delta-0} = \beta_0 \frac{\partial V}{\partial x}\bigg|_{x=\delta+0}. \qquad (7)$$

2.2 Description of Temperature Field in the Fin

The rectangular fin of length l occupies the domain $\{x \in [\delta, \delta + l], y \in [0, b], z \in [0, w]\}$ and the temperature field $V(x, y, z)$ fulfills the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0. \qquad (8)$$

We have following boundary conditions for the fin:

$$\frac{\partial V}{\partial x} + \beta V = 0,\ x = \delta + l, \qquad (9)$$

$$\frac{\partial V}{\partial y} + \beta V = 0,\ y = b, \qquad (10)$$

$$\frac{\partial V}{\partial z} + \beta V = 0,\ z = w, \qquad (11)$$

$$\frac{\partial V}{\partial x}\bigg|_{x=0} = \frac{\partial V}{\partial y}\bigg|_{y=0} = 0, \qquad (12)$$

$$\frac{\partial V}{\partial z}\bigg|_{z=0} = 0. \qquad (13)$$

Let's mention, that almost all of the authors negligible the heat transfer trough flank surface $z = w$. We assume that this heat transfer is proportional to the temperature excess between the wall/fin and the surrounding medium and are given by second boundary condition (5) and condition (11).

3 Approximate Solution of 3-D Problem

We will use the original method of conservative averaging.

3.1 Reduction of the 3-D Problem to the 2-D Problem

Similarly as in our previous papers [4],[5] we will use our original method of conservative averaging and approximate the 3-D temperature field $V(x, y, z)$ for the fin in following form:

$$V(x, y, z) = h_0(x, y) + (e^{\sigma z} - 1)h_1(x, y) + (1 - e^{-\sigma z})h_2(x, y),\ \sigma = w^{-1} \qquad (14)$$

with unknown functions $h_i(x, y),\ i = 0,1,2.$ For this purpose we introduce the integral average value of function $V(x, y, z)$ in the z - direction:

$$U(x, y) = \sigma \int_0^w V(x, y, z)dz. \qquad (15)$$

This equality together with two boundary conditions (at $z = 0$ and $z = w$) allow us to exclude all unknown functions $h_i(x, y)$ from the representation (13). The boundary condition (13) gives the equality:

$$h_2(x, y) = -h_1(x, y).$$

The substitution of representation (14) in (15) gives expression:

$$h_1(x, y) = \frac{U(x, y) - h_0(x, y)}{2(\sinh(1) - 1)} \qquad (16)$$

and representation (14) takes form:

$$V(x, y, z) = \frac{\cosh(\sigma z) - 1}{\sinh(1) - 1} U(x, y) + \frac{\sinh(1) - \cosh(\sigma z)}{\sinh(1) - 1} h_0(x, y).$$

Finally, by the use of the boundary condition (11) we can exclude $h_0(x, y)$ from last expression and represent the 3-D solution $V(x, y, z)$ for the fin in following form:

$$V(x, y, z) = U(x, y)\Psi(z). \qquad (17)$$

It is easy to check that the function $\Psi(z)$ looks like

$$\Psi(z) = \frac{\sinh(1) + \beta w[\cosh(1) - \cosh(\sigma z)]}{\sinh(1) + \beta w[\cosh(1) - \sinh(1)]}. \qquad (18)$$

The second stage for the method of conservative averaging is the transforming of the differential equation (8) for the function $V(x, y, z)$ to the differential equation for the function $U(x, y)$. To realize this goal we integrate the main differential equation (8) in the z - direction:

$$\frac{\partial^2 U(x, y)}{\partial x^2} + \frac{\partial^2 U(x, y)}{\partial y^2} + \frac{\partial V}{\partial z}\bigg|_{z=0}^{z=w} = 0. \qquad (19)$$

Expressing from the boundary conditions (11) and (13) the first derivatives of the function $V(x, y, z)$ at $z = 0$ and $z = w$ trough the function $U(x, y)$ we finally obtain following partial differential equation for two-dimensional temperature field $U(x, y)$ in the fin:

$$\frac{\partial^2 U(x, y)}{\partial x^2} + \frac{\partial^2 U(x, y)}{\partial y^2} - \bar{\mu}^2 U(x, y) = 0. \qquad (20)$$

Here $\bar{\mu}^2 = \beta w^{-1}\Psi(w)$.

The same procedure for the wall gives the representation:

$$V_0(x,y,z) = U_0(x,y)\Psi(z). \quad (21)$$

Here $U_0(x,y)$ again (similar with equality (15)) is the integral average value of function $V_0(x,y,z)$ in the z - direction:

$$U_0(x,y) = \sigma\int_0^w V_0(x,y,z)dz.$$

Finally we obtain following partial differential equation for two-dimensional temperature field $U_0(x,y)$ for the wall:

$$\frac{\partial^2 U_0(x,y)}{\partial x^2} + \frac{\partial^2 U_0(x,y)}{\partial y^2} - \bar{\mu}^2 U_0(x,y) = 0. \quad (22)$$

3.2 Solution of the 2-D Problem

From here again we can use the conservative averaging method for 2-D as in our previous papers [4],[5], but with one distinction: in previous subsection we have obtained Helmholtz equations (20),(22) instead of Laplace equation. However this difference doesn't make additional difficulties for the utilization of the conservative averaging method. This is why we will briefly describe further steps of solution of the equations (20),(22) together with boundary conditions (2)-(4),(9),(10),(12) and conjugations conditions (6),(7).

We will approximate the 2-D temperature field $U(x,y)$ in the fin in the form similar to the representation (14):

$$U(x,y) = f_0(x) + (e^{\rho y} - 1) f_1(x) + (1 - e^{-\rho y}) f_2(x),$$
$$\rho = b^{-1}.$$

We introduce the second integral average value of function $V(x,y,z)$, but in the y - direction now (see (15)):

$$u(x) = \rho\int_0^b U(x,y)dy. \quad (23)$$

Repeating all steps as in subsection 3.1 we finally obtain the solution for the fin in the form similar to approximation (17):

$$U(x,y) = u(x)\Phi(y). \quad (24)$$

Here the expression for the function $\Phi(y)$ is similar to expression (18) for the function $\Psi(z)$:

$$\Phi(y) = \frac{\sinh(1) + \beta b[\cosh(1) - \cosh(\rho y)]}{\sinh(1) + \beta b[\cosh(1) - \sinh(1)]}. \quad (25)$$

It consequently follows from (17) and (24):

$$V(x,y,z) = u(x)\Phi(y)\Psi(z). \quad (26)$$

Now we integrate the equation (20) in the y - direction (similar as in equation (19)) and use following boundary conditions (see the boundary condition (10) and second of boundary conditions (12)):

$$\frac{\partial U}{\partial y} + \beta U = 0,\ y = b,\ \left.\frac{\partial U}{\partial y}\right|_{y=0} = 0.$$

Then we finally obtain that the function $u(x)$ is the solution of following ordinary differential equation:

$$\frac{d^2u}{dx^2} - \mu_0^2 u = 0,\ \mu_0^2 = \beta\rho\Phi(b) + \bar{\mu}^2.$$

The solution of this ordinary differential equation for $x \in (\delta, \delta + l)$ together with the boundary condition (corollary from the boundary condition (9) after its integration in y and z directions)

$$\left.\left[\frac{du(x)}{dx} + \beta u(x)\right]\right|_{x=\delta+l} = 0$$

gives:

$$u(x) = C_1(\mu_1 \exp(\mu_0 x) + \exp(-\mu_0 x)). \quad (27)$$

Here C_1 is temporarily unknown constant and

$$\mu_1 = \frac{\mu_0 - \beta}{\mu_0 + \beta}\exp(-2\mu_0(\delta + l)).$$

We act almost equally for the wall and approximate the 2-D temperature field $U_0(x,y)$ for the wall in $x-$direction:

$$U_0(x,y) = g_0(y) + [e^{d(\delta - x)} - 1]g_1(y) +$$
$$[1 - e^{d(x-\delta)}]g_2(y),\ d = \delta^{-1}, \quad (28)$$

Now we introduce the integral average value of function $U_0(x,y)$ in x direction:

$$u_0(y) = d\int_0^\delta U_0(x,y)dx. \quad (29)$$

The equality (29) and boundary condition (2) allow us to express $g_i(y), = 1,2$ in the form:

$$g_i(y) = (-1)^i[b_i u_0(y) - a_i g_0(y) + d_i], i = 1,2. \quad (30)$$

Here $a_i = A_i K_1^{-1},\ b_i = B_i K_1^{-1},\ d_i = D_i K_1^{-1},\ i = 1,2$,

$$K_1 = e^{-1}[2 + \beta_0^0\delta(e-1)(3-e)],$$
$$A_1 = e^{-1}(1 + \beta_0^0\delta(e-2)),\ A_2 = e + \beta_0^0\delta,$$
$$B_1 = e^{-1}(1 + \beta_0^0\delta(e-1)),\ B_2 = e\ + \beta_0^0\delta(e-1)),$$
$$D_1 = e^{-1}\beta_0^0\delta,\ D_2 = \beta_0^0\delta(e-2).$$

Therefore we can rewrite the expression (28) for $U_0(x,y)$ in the form:

$$U_0(x,y)=[1+(e^{d(\delta-x)}-1)a_1-(1-e^{d(x-\delta)})a_2]\times g_0(y)+[(1-e^{d(x-\delta)})b_2-(e^{d(\delta-x)}-1)b_1]u_0(y)+(e^{d(\delta-x)}-1)d_1-(1-e^{d(x-\delta)})d_2. \quad (31)$$

By the use of boundary condition (3) for the upper part of the wall b<y<1 we get the connection between functions $g_0(y)$ and $u_0(y)$:

$$g_0(y)=b_0u_0(y)-d_0. \quad (32)$$

In expression (32):

$b_0 = B_0K_0^{-1},\ d_0 = D_0K_0^{-1},\ B_0 = B_2 - B_1,$

$D_0 = D_2 - D_1,\ K_0 = A_2 - A_1 + \beta_0\delta K_1.$

By integrating the differential equation (22) in the $y-$ direction and by using boundary conditions (2),(3) and second of the conditions (4) we receive the following equation:

$$\frac{d^2u_0}{dy^2}-\tilde{k}^2u_0+\Theta_2=0,\ b<y<1,\ u'(1)=0, \quad (33)$$

$\tilde{k}^2=2\delta^{-2}K_0^{-1}[(\beta_0+\beta_0^0)\delta\sinh(1)$
$+2\beta_0\beta_0^0\delta^2(\cosh(1)-1)]+\bar{\mu}^2,$

$\Theta_2=\delta^{-2}[(d_1-a_1d_0)(e-1)+(d_2-a_2d_0)(1-e^{-1})].$

The solution of the problem (33) is:

$$u_0(y)=C_2\cosh(\tilde{k}(1-y))+U_2,U_2=\tilde{k}^{-2}\Theta_2. \quad (34)$$

Here C_2 is unknown constant again.

For lower part $0<y<b$ of the wall we use conditions (6),(7) and we finally get the following equation:

$$\frac{d^2u_0}{dy^2}-\tilde{\lambda}^2u_0+C_1\Theta_{3,0}-C_1\Theta_{3,1}\cosh(\rho y)+D_3=0.$$

Solution of this equation with boundary condition $\frac{du_0(0)}{dy}=0$ for general case $\rho\neq\tilde{\lambda}$ looks as follow:

$$u_0(y)=C_3\cosh(\tilde{\lambda}y)+C_1(\vartheta_{3,0}+\vartheta_{3,1}\cosh(\rho y))+d_3. \quad (35)$$

Here the parameters are (C_3 is temporarily unknown constant again):

$\tilde{\lambda}^2=\frac{\beta_0^0(e-1)^2}{e\delta K_1}+\bar{\mu}^2, D_3=\frac{2\tilde{\lambda}^2}{(e-1)^2},$

$\Theta_{3,i}=\Theta_3\Phi_i,\ i=0,1,\ \ d_3=\frac{D_3}{\tilde{\lambda}^2},\ \ \vartheta_{3,0}=\frac{\Theta_{3,0}}{\tilde{\lambda}^2},$

$\Theta_3=\frac{(A_1e-A_2e^{-1})(1+\mu_1)}{\delta^2K_1}-\frac{\beta_0\tilde{\mu}}{\beta\delta}(1-\mu_1),$

$\vartheta_{3,1}=\Theta_{3,1}/(\rho^2-\tilde{\lambda}^2),\ \Phi_0=(\sinh(1)(\beta b)^{-1}$
$+\cosh(1))\Phi_1, \Phi_1=(\sinh(1)(\beta b)^{-1}+\cosh(1)$
$-\sinh(1))^{-1}.$

To complete the solving of this 3-D problem by the method of conservative averaging we should determine the three unknown constants $C_i,\ i=1,2,3$ from the equations (27),(34) and (35). For this purpose we formulate three natural requirements [4],[5]. The first of them is the continuity of temperatures in the certain point - $x=\delta, y=b$ - in the fin and upper part of the wall. From equalities (27),(32) and (34) we receive:

$C_1(\mu_1\exp(\tilde{\mu}\delta)+\exp(-\tilde{\mu}\delta))\Phi(b)$
$-C_2b_0\cosh(\tilde{k}(1-b))=U_2b_0-d_0.$

As the next two requirements we assume the continuity of temperature and heat flux on the line $0<x<\delta, y=b$ between the upper and lower parts of the wall. We finally get from (34) and (35):

$C_2\cosh(\tilde{k}(1-b))-C_1[\vartheta_{3,0}+\vartheta_{3,1}\cosh(1)]$
$-C_3\cosh(\tilde{\lambda}b)=d_3-U_2,$

$C_1\vartheta_{3,1}\rho\sinh(1)+C_2\tilde{k}\sinh(\tilde{k}(1-b))$
$+C_3\tilde{\lambda}\sinh(\tilde{\lambda}b)=0.$

One can easy prove that this system of three linear algebraic equations has only one solution. Some numerical results and their analysis for 2-D solution were presented in paper [7].

3.3 Special Cases of the 3-D Solution

We will have the first special case, if we assume the independence of 3-D solution according to third variable - argument z. Then equality (15) reduces to trivial identity:

$V(x,y,z)\equiv U(x,y)$, similarly

$V_0(x,y,z)=U_0(x,y)$.

Parameter $\bar{\mu}$ reduces in this case to the expression $\bar{\mu}^2=\beta w^{-1}$.

The next special case we will have, if we assume the insulation of flank surface $z=w$. In this case we have $\bar{\mu}^2=0$ and 3-D solution reduces to our previous 2-D solution [4],[5].

3.4 Classical 1-D Solution as the Simple Case of the 3-D Solution

The well known 1-D statement for the periodical system with rectangular fin from papers [8],[9] looks as follow:

$$U_0''(x) = 0, \quad (36)$$

$$bU''(x) - \beta U(x) = 0, \quad x \in (\delta, \delta + l), \quad (37)$$

$$U_0'(x) + \beta_0^0 (1 - U_0) = 0, \quad x = 0, \quad (38)$$

$$U'(x) + \beta U = 0, \; x = \delta + l, \quad (39)$$

$$U_0\big|_{x=\delta-0} = U\big|_{x=\delta+0}, \quad (40)$$

$$\beta\left[U_0'(x) + \beta_0(1-b)U_0\right]\Big|_{x=\delta-0} = \beta_0 b U'(x)\Big|_{x=\delta+0}. \quad (41)$$

The well known solution of problem (36)-(41) can be written in following form (see [8], [9]):

$$U_0(x) = C_1^{(0)} x + C_0^{(0)}, U(x) = C_1 \exp(\mu x) + C_2 \exp(-\mu x), \mu = \beta / b. \quad (42)$$

Here the four unknown constants can be easy determined from the four boundary and conjugations conditions (38)-(41).

This solution can be easy obtained as special case of our 3-D solution by the integrating in the y-direction, if we will make additionally two simplifications:

1) we assume independence of both temperatures from two arguments - y and z , i.e., we suppose, that $V(x, y, z) \equiv U(x)$, and $V_0(x, y, z) \equiv U_0(x)$;

2) we assume that parameter $\bar{\mu}^2 = 0$. It means that we have made the assumption of insulation condition instead of the heat exchange with the surrounding medium on the surface $z = w$.

We can easily propose a more advanced solution (in comparison with "classical" solution (42)) , if we presume the independence of 3-D solution with respect to second and third variables, however we will take in account heat exchange with surrounding medium according to boundary conditions (5) and (11) by the method of conservative averaging. Finally, instead of solution (42) we receive the solution in the form of:

$$U_0(x) = \tilde{C}_1^{(0)} \exp(\mu^{(0)} x) + \tilde{C}_2^{(0)} \exp(-\mu^{(0)} x),$$
$$U(x) = \tilde{C}_1 \exp(\mu^{(1)} x) + \tilde{C}_2 \exp(-\mu^{(1)} x), \quad (43)$$
$$\mu^{(0)} = \beta / w, \mu^{(1)} = \beta(1/b + 1/w).$$

This new and simple solution has a form which permits to estimate easily the cases when it is possible to use the solution (42) instead of solution (43). These conditions are:

$\mu^{(0)} \delta << 1, b << w$.

4 Conclusion

We have constructed the approximate three dimensional analytical solution for a periodical system with rectangular fin for the case when the wall and the fin consist of materials which have different thermal properties.

We have shown that it is possible to get previously acquired two dimensional and "classical" one dimensional solution and some of its extendings from general approximate 3-D solution, which is obtained in this paper, after simplifying some of the assumptions.

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