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Approximate analytical three-dimensional solution for periodical system with rectangular fin, Part 2

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Abstract: - In this paper the approximate transient three dimensional analytical solution for the element of periodical system with rectangular fin is obtained by the original method of conservative averaging. The solution has a form of three 1-D heat equations with source terms which are linearly dependent of temperature. All three equations are connected by some natural junction conditions.

Key-Words: - heat transfer, rectangular fin, transient process, three-dimensional, analytical solution, conservative averaging.

1 Introduction

A great number of different engineering branches are concerned with rapid heat energy transitions. In the construction of various types of efficient heat transfer equipment to the so-called prime surface are supplemented an additional surfaces, e.g., a rectangular fin. Such heat transfer equipment is related to refrigerators, radiators, engines and microelectronics, etc. The traditional mathematical description of heat flow between a source and a sink very often is bounded by the so-called Murray-Gardner's hypotheses [1], [2]. One of these hypotheses is an assumption that the heat flow at any point in the prime surface and in the fin remains constant with time.

In this second part of our paper we will study the transient heat transfer in one element of 3-D system with rectangular fin. We transform the initial 3-D problem to the system of connected three one dimensional partial differential equations of parabolic type (heat equations with constant coefficients and linear sink term). For the solution of this system we propose the finite difference method.

2 Mathematical Formulation of 3-D Problem

We will start with three-dimensional formulation of transient problem for one element of periodical system with rectangular fin. Similar mathematical formulation for 2-D case was given in our paper [3]. The 2-D steady-state problem was considered in our paper [4].

2.1 Description of Temperature Field in the Wall

We will use the same dimensionless arguments and parameters as in Part 1 of this paper with only one addition: $a^2 = k/c, (a^2 = k_0/c_0)$ is temperature conductivity coefficient for the fin (wall), where $c(c_0)$ is specific heat capacity of the fin (wall).

The wall - one element of periodical system with rectangular fin - is placed in the 3-D domain $\{x \in [0,\delta], y \in [0,1], z \in [0,w]\}$ and there the temperature field $V_0(x,y,z,t)$ is described with the heat equation:

$$\frac{\partial^2 V_0}{\partial x^2} + \frac{\partial^2 V_0}{\partial y^2} + \frac{\partial^2 V_0}{\partial z^2} = a_0^{-2} \frac{\partial V_0}{\partial t}. \tag{1}$$

We add to this main partial differential equation following boundary conditions:

$$\frac{\partial V_0}{\partial x} + \beta_0^0 (1 - V_0) = 0,\ x = 0, \tag{2}$$

$$\frac{\partial V_0}{\partial x} + \beta_0 V_0 = 0,\ x = \delta, y \in (b,1), \tag{3}$$

$$\left.\frac{\partial V_0}{\partial y}\right|_{y=0} = \left.\frac{\partial V_0}{\partial y}\right|_{y=1} = 0, \tag{4}$$

$$\left.\frac{\partial V_0}{\partial z}\right|_{z=0} = 0,\quad \frac{\partial V_0}{\partial z} + \beta_0 V_0 = 0,\ z = w. \tag{5}$$

We assume the classical conjugations conditions on the surface between the wall and the fin:

$$V_0\big|_{x=\delta-0} = V\big|_{x=\delta+0}, \tag{6}$$

$$\beta \frac{\partial V_0}{\partial x}\bigg|_{x=\delta-0} = \beta_0 \frac{\partial V}{\partial x}\bigg|_{x=\delta+0}. \tag{7}$$

Finally we add the initial condition in form:

$$V_0(x,y,z,t)\big|_{t=0} = V_0^0(x,y,z). \tag{8}$$

As one can see the difference between this formulation of the problem and the one, which is given in the first part of the paper is in the main differential equation (1) and in initial condition (8).

2.2 Description of Temperature Field in the Fin

The rectangular fin occupied the domain $\{x\in[\delta,\delta+l], y\in[0,b], z\in[0,w]\}$ and there the temperature field $V(x,y,z,t)$ fulfills the heat equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = a^{-2}\frac{\partial V}{\partial t}. \tag{9}$$

We have following boundary conditions for the fin (they are identical with boundary conditions given in first part of this paper):

$$\frac{\partial V}{\partial x} + \beta V = 0,\ x = \delta + l, \tag{10}$$

$$\frac{\partial V}{\partial y} + \beta V = 0,\ y = b, \tag{11}$$

$$\frac{\partial V}{\partial z} + \beta V = 0,\ z = w, \tag{12}$$

$$\frac{\partial V}{\partial x}\bigg|_{x=0} = \frac{\partial V}{\partial y}\bigg|_{y=0} = 0, \tag{13}$$

$$\frac{\partial V}{\partial z}\bigg|_{z=0} = 0 \tag{14}$$

and initial condition

$$V(x,y,z,t)\big|_{t=0} = V^0(x,y,z). \tag{15}$$

3 Approximate Transforming the 3-D Formulation to Transient 1-D Problem

Again we will use our original method of conservative averaging. This approximate transition from 3-D statement to the 2-D formulation is very similar with the steady-state case.

3.1 Reduction of the 3-D Problem to the 2-D Problem

We will approximate the transient 3-D temperature field $V(x,y,z,t)$ for the fin in following form:

$$\begin{aligned} V(x,y,z,t) &= h_0(x,y,t) + (e^{\sigma z}-1)h_1(x,y,t) \\ &+ (1-e^{-\sigma z})h_2(x,y,t),\ \sigma = w^{-1} \end{aligned} \tag{16}$$

Here the three functions $h_i(x,y,t),\ i=0,1,2$ are unknown. Again we introduce the integral average value of function $V(x,y,z,t)$ in the z - direction:

$$U(x,y,t) = \sigma\int_0^w V(x,y,z,t)dz. \tag{17}$$

The last equality together with two boundary conditions (at $z=0$ and $z=w$) allow us to exclude all $h_i(x,y)$ from the representation (16). Finally we can represent the 3-D solution $V(x,y,z,t)$ for the fin in the form:

$$V(x,y,z,t) = U(x,y,t)\Psi(z). \tag{18}$$

Here the function $\Psi(z)$ has the expression:

$$\Psi(z) = \frac{\sinh(1) + \beta w[\cosh(1) - \cosh(\sigma z)]}{\sinh(1) + \beta w[\cosh(1) - \sinh(1)]}. \tag{19}$$

Now we transform the differential equation (9) to the differential equation for the function $U(x,y,t)$: we integrate the equation (9) in the z - direction. Finally we obtain following equation for the 2-D temperature field $U(x,y,t)$ in the fin:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} - \bar{\mu}^2 U = a^{-2}\frac{\partial U}{\partial t}, \tag{20}$$

$$\bar{\mu}^2 = \beta w^{-1}\Psi(w).$$

The same procedure for the wall gives identical representation:

$$V_0(x,y,z,t) = U_0(x,y,t)\Psi(z). \tag{21}$$

Here $U_0(x,y)$ again is the integral average value of function $V_0(x,y,z,t)$ in the z - direction:

$$U_0(x,y,t) = \sigma\int_0^w V_0(x,y,z,t)dz. \tag{22}$$

The heat equation for two-dimensional temperature field $U_0(x,y,t)$ for the wall takes the form identical to the equation (20) for the fin:

$$\frac{\partial^2 U_0}{\partial x^2} + \frac{\partial^2 U_0}{\partial y^2} - \bar{\mu}^2 U_0 = a_0^{-2}\frac{\partial U_0}{\partial t}. \tag{23}$$

We need to add the averaged in the z - direction initial conditions to the heat equations (20) and (23):

$$\begin{aligned} U(x,y,t)\big|_{t=0} &= U^0(x,y), \\ U_0(x,y,t)\big|_{t=0} &= U_0^0(x,y). \end{aligned} \tag{24}$$

3.2 Reduction of the 2-D Problem to the 1-D Problem

From here again we can use the conservative averaging method for 2-D problems with the previous boundary conditions rewritten for the 2-D temperatures $U(x,y,t)$ and $U_0(x,y,t)$. This can be done in the way which is practically identical as in first part of this paper. We will approximate the 2-D temperature field $U(x,y,t)$ in the fin in the form:

$$U(x,y,t) = f_0(x,t) + (e^{\rho y} - 1) f_1(x,t) + (1 - e^{-\rho y}) f_2(x,t), \rho = b^{-1}. \tag{25}$$

The second integral average value of function $V(x,y,z,t)$ is defined as follow:

$$u(x,t) = \rho \int_0^b U(x,y,t) dy. \tag{26}$$

We repeat all the steps as in part 1 and we finally get:

$$U(x,y,t) = u(x,t)\Phi(y). \tag{27}$$

Here the expression for the function $\Phi(y)$ is the same as in part 1:

$$\Phi(y) = \frac{\sinh(1) + \beta b[\cosh(1) - \cosh(\rho y)]}{\sinh(1) + \beta b[\cosh(1) - \sinh(1)]}. \tag{28}$$

Finally we have from (18) and (27) the approximate representation for the three dimensional temperature field in the fin:

$$V(x,y,z,t) = u(x,t)\Phi(y)\Psi(z). \tag{29}$$

Now we can integrate the differential equation (20) in the y-direction and use following boundary conditions (see the boundary condition (11) and second of boundary conditions (13)):

$$\frac{\partial U}{\partial y} + \beta U = 0,\ y = b,\ \left.\frac{\partial U}{\partial y}\right|_{y=0} = 0.$$

Then we obtain that the function $u(x,t)$ is the solution of following 1-D parabolic type partial differential equation:

$$\frac{\partial^2 u}{\partial x^2} - \tilde{\beta}^2 u = a^{-2}\frac{\partial u}{\partial t},\ \tilde{\beta}^2 = \beta\rho\Phi(b) + \bar{\mu}^2. \tag{30}$$

This differential equation must be solved for $x \in (\delta, \delta + l)$ together with the boundary condition

$$\left.\left[\frac{\partial u(x,t)}{\partial x} + \beta u(x,t)\right]\right|_{x=\delta+l} = 0 \tag{31}$$

and with the initial condition

$$u(x,t)\big|_{t=0} = u^0(x). \tag{32}$$

Both these conditions are the corollary from the conditions (8) and (15) after they are integrated (averaged) in y - and z - directions.

We act almost equally for the wall and approximate the 2-D temperature field $U_0(x,y,t)$ for the wall in x - direction:

$$U_0(x,y,t) = g_0(y,t) + [e^{d(\delta - x)} - 1] \times g_1(y,t) + [1 - e^{d(x-\delta)}] g_2(y,t),\ d = \delta^{-1}. \tag{33}$$

Now we introduce again the integral average value of function $U_0(x,y,t)$ in x - direction:

$$u_0(y,t) = d\int_0^\delta U_0(x,y,t)dx. \tag{34}$$

The equality (34) and boundary condition (2) allow us to express $g_i(y), = 1,2$ in the form:

$$g_i(y,t) = (-1)^i [b_i u_0(y,t) - a_i g_0(y,t) + d_i],\ i = 1,2. \tag{35}$$

Here all the coefficients have the same expressions as in part 1:

$$a_i = A_i K_1^{-1},\ b_i = B_i K_1^{-1},\ d_i = D_i K_1^{-1},\ i = 1,2,$$
$$K_1 = e^{-1}[2 + \beta_0^0 \delta(e-1)(3-e)],$$
$$A_1 = e^{-1}(1 + \beta_0^0\delta(e-2)),\ A_2 = e + \beta_0^0\delta,$$
$$B_1 = e^{-1}(1 + \beta_0^0\delta(e-1)),\ B_2 = e + \beta_0^0\delta(e-1)),$$
$$D_1 = e^{-1}\beta_0^0\delta,\ D_2 = \beta_0^0\delta(e-2).$$

Therefore we can rewrite the expression for $U_0(x,y,t)$ in the form:

$$U_0(x,y,t) = [1 + (e^{d(\delta-x)} - 1)a_1 - (1 - e^{d(x-\delta)})a_2] \times g_0(y,t) + [(1 - e^{d(x-\delta)})b_2 - (e^{d(\delta-x)} - 1)b_1] \times u_0(y,t) + (e^{d(\delta-x)} - 1)d_1 - (1 - e^{d(x-\delta)})d_2.$$

By the use of boundary condition (3) for the upper part of the wall b<y<1 we get the connection between functions $g_0(y,t)$ and $u_0(y,t)$:

$$g_0(y,t) = b_0 u_0(y,t) - d_0. \tag{36}$$

In expression (36):

$$b_0 = B_0 K_0^{-1},\ d_0 = D_0 K_0^{-1},\ B_0 = B_2 - B_1,$$
$$D_0 = D_2 - D_1,\ K_0 = A_2 - A_1 + \beta_0 \delta K_1.$$

By integrating the differential equation (23) in the $y-$ direction and by using boundary conditions (2), (3) and second of the conditions (4) we receive the following equation:

$$\frac{\partial^2 u_0}{\partial y^2} - \tilde{k}^2 u_0 + \Theta_2 = a_0^{-2}\frac{\partial u_0}{\partial t},\ b < y < 1, \tag{37}$$

$$\frac{\partial u_0(1,t)}{\partial y} = 0, \tag{38}$$

$$\tilde{k}^2 = 2\delta^{-2}[(\beta_0 + \beta_0^0)\delta\sinh(1) + 2\beta_0\beta_0^0\delta^2(\cosh(1) - 1)]/K_0 + \bar{\mu}^2,$$
$$\Theta_2 = \delta^{-2}[(d_1 - a_1 d_0)(e-1) + (d_2 - a_2 d_0)(1 - e^{-1})].$$

For lower part $0 < y < b$ of the wall we use conditions (6), (7) and we finally get the following equation:

$$\frac{\partial^2 u_0}{\partial y^2} - \tilde{\lambda} u_0 + \Phi(y)\left[E_0 u + E_1 \frac{\partial u}{\partial x}\right]\Bigg|_{x=\delta} \qquad (39)$$

$$+D_3 = a_0^{-2}\frac{\partial u_0}{\partial t}, 0 < y < b.$$

Here

$$\tilde{\lambda} = \frac{\beta_0^0 (e-1)^2}{e\delta K_1} + \bar{\mu}^2, D_3 = \frac{2\tilde{\lambda}}{(e-1)^2},\ d_3 = \frac{D_3}{\tilde{\lambda}},$$

$$E_0 = \frac{\beta_0^0 (e^2 - 2e - 1)}{e\delta K_1}, E_1 = \frac{\beta_0}{\beta\delta}.$$

To differential equation (39) we must add averaged in x- and z-directions boundary condition (4) in form: $\frac{\partial u_0(0,t)}{\partial y} = 0$ (40)

and initial condition (24), which is integrated in x-direction:

$$u_0(y,t)\big|_{t=0} = u_0^0(y). \qquad (41)$$

This initial condition is suitable for both the differential equations (37) and (39).

This problem for three partial differential equations (30), (37) and (39) with mentioned boundary and initial conditions we solve with the finite difference method, e.g. [5],[6]. But we haven't conditions for the junction the solutions of the differential equations (37), (39) on the line $y = b$. There is simple outcome: we assume the continuity of function (temperature) $u_0(y,t)$ and heat flux in the corner point $\{x = \delta, y = b\}$. For the junction of the fin and wall temperatures in this point $\{x = \delta, y = b\}$ we can use consequences from the equalities (6), (18), (21), (27) and (36):

$$u(\delta,t)\Phi(b) = b_0 u_0(b,t) - d_0. \qquad (42)$$

For the solution of these differential equations together with boundary, junction and initial conditions we use the classical weighted three point difference schemes of second order of accuracy according the space variables. To achieve the second order of accuracy for the approximation of boundary conditions we apply the method of using main differential equation on the boundary [6]. When we try to approximate in ordinary way the differential equation (39) we have untypical situation because this equation includes the terms with temperature and heat flux from the fin. This difficulty was solved in [3] by generalizing Gauss elimination method for three diagonal matrix.

4 Conclusion

We have reduced 3-D transient heat transfer mathematical model for one element of the periodical system with rectangular fin to approximate 1-D problem for three partial differential equations with constant coefficients. They are connected together with natural junction conditions. Generalization of classical finite difference schemes for their solution was proposed.

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