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On the stochastic stability of the discrete time jump linear system[1]

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Abstract: This paper is concerned with problems of mean square stability of discrete and continuous time linear systems parameters of which are dependent on finite-state Markov processes which are directly observed. For this problem new sufficient conditions are obtained. These conditions are simpler to check than the known ones.

Key-Words: Stochastic stability; mean square stability; systems with Markovian jumps.

1. Introduction

Systems subject to abrupt changes or with uncertain dynamics can be naturally modelled as jump linear systems. Because of their applications in fields such as tracking, fault-tolerant control [9], flexible manufacturing processes [11], power systems [10], such systems have drawn extensive attention. Consider discrete-time linear system with Markovian jumps, modelled by

$$x(k) = A(r(k))x(k), \qquad (1)$$

where $r(k)$ is a Markov chain taking values in a finite set $S = \{1,2,\ldots,s\}$, characterised by constant probability matrix $P = (p_{ij})_{i,j\in S}$, where

$$p_{ij} = \Pr(r(k+1) = j \mid r(k) = i)$$

and initial distribution $p = [p_j]_{j\in S}$, $p_j = \Pr(r(0) = j)$. $A(j)$, $j \in S$ are matrices of size $n \times n$. Solution of (1) with initial condition $x(0) = x_0 \in R^n$ is denoted by $x(k, x_0, p)$. Denote by $X \otimes Y$ Kronecker product of matrices *X* and *Y*, and by I_m identity matrix of size *m*. Set of all eigenvalues of a square matrix *Z* is denoted $s(Z)$. In the case of symmetric matrix *Z* we assume that the elements $l_1(Z),\ldots,l_n(Z)$ of $s(Z)$ are numbered such that $l_1(Z) \geq \ldots \geq l_n(Z)$. By X' we denote transposition of matrix *X*.

Different concepts of stability of system (1) have been investigated in many papers, see [1]-[5]. In this paper we consider one of them namely mean square stability. Formal definition is given below.

Definition ([1], [2]) The system (1) is called mean square stable, if for all initial conditions $x_0 \in R^n$ and all initial distributions p, we have

$$\lim_{k\to\infty} E\|x(k, x_0, p)\|^2 = 0.$$

Condition for mean square stability in terms of solutions of certain type of coupled Lyapunov equation has been given in [2]. Other conditions for this type of stability has been proved in [5] and they are presented now.

Theorem 1 [5] System (1) is mean square stable if and only if

$$\max\{|l| : l \in s(F)\} < 1, \qquad (2)$$

where $F = (P' \otimes I_{n^2})\, diag[A_i \otimes A_i]_{i=1,\ldots,s}$.

In this paper we simplify the condition (2) to obtain a sufficient condition for mean square stability which is less numerically involved.

2. Main result

This theorem completely solves the problem of second moment stability. However, condition (2) requires calculation of eigenvalues of the matrix which is of very high dimension (sn^2 by sn^2). Therefore one may be interested in sufficient conditions which are given in terms of matrices of

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lower sizes. The next theorem gives such a condition.

Theorem 2 The discrete time jump linear system (1) is stable if

$$l_1(PP')\sum_{i=1}^{s} l_1^{\,2}\left(A_i A_i'\right) < 1. \qquad (3)$$

In the proof we need the following two lemmas.

Lemma 1 [6] Let *X, Y, Z* be square *n* by *n* matrices with *X'=X, Y' =Y.* Then the following inequalities hold

$$l_1(X+Y) \le l_1(X) + l_1(Y), \qquad (4)$$

$$\max_{i=1,\dots,n}\{|l| : l \in s(Z)\} \le \sqrt{l_1(ZZ')}. \qquad (5)$$

Lemma 2 [7] If A and B are square matrices and $p(x,y) = \sum_{k,l=0}^{p} c_{kl} x^k y^l$ is a complex polynomial of two variables, then

$$s\left(\sum_{k,l=0}^{p} c_{kl} A^k \otimes B^l\right) = \{p(x,y) : x \in s(A), y \in s(B)\}\cdot$$

Proof of the Theorem 2 From the properties of Kronecker product we have

$$\left(A_i A_i'\right)\otimes\left(A_i A_i'\right) = \left(A_i \otimes A_i\right)\left(A_i \otimes A_i\right)'.$$

This together with definition of *F* implies

$$FF' = (PP')\otimes\left(\sum_{i=1}^{s}\left(\left(A_i A_i'\right)\otimes\left(A_i A_i'\right)\right)\right).$$

By Lemma 2 we have

$$s(FF') = \left\{ l_k(PP')\, l_l\left(\sum_{i=1}^{s}\left(\left(A_i A_i'\right)\otimes\left(A_i A_i'\right)\right)\right) : k = 1,..,n,\ l = 1,\dots,n^2 \right\}. \qquad (6)$$

Applying (4) gives

$$l_1\left(\sum_{i=1}^{s}\left(\left(A_i A_i'\right)\otimes\left(A_i A_i'\right)\right)\right) \le \sum_{i=1}^{s} l_1\left(\left(A_i A_i'\right)\otimes\left(A_i A_i'\right)\right). \qquad (7)$$

Using again Lemma 2 we get

$$l_1\left(\left(A_i A_i'\right)\otimes\left(A_i A_i'\right)\right) = l_1^2\left(A_i A_i'\right). \qquad (8)$$

Combining (6), (7) and (8) we can bound $l_1(FF')$ as follows

$$l_1(FF') \le l_1(PP')\sum_{i=1}^{s} l_1^2\left(A_i A_i'\right)\cdot \qquad (9)$$

Now from (5) and (9) we obtain

$$\max\{|l| : l \in s(F)\} \le \sqrt{l_1(FF')} \le \sqrt{l_1(PP')\sum_{i=1}^{s} l_1^2\left(A_i A_i'\right)}$$

and the proof follows from Theorem 1.

3. Numerical example

Consider system (1) with:

$$P = \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.3 & 0.4 \\ 0.1 & 0.5 & 0.4 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix},\ A_3 = \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix}$$

and suppose that $a \ge b > 0$. We are interested in values a, and b for which system is mean square stable. The use of Theorem 1 requires analysis of eigenvalues of 12 by 12 matrix. This approach is very involved. To apply Theorem 2 we calculate

$$l_1(PP') \approx 1.0833,\ l_1(A_1A_1') = a^2,$$
$$l_1(A_2A_2') = a^2,\ l_1(A_3A_3') = b^2.$$

According to (3) we know that the system is stable if

$$a^2 + 2b^2 < 0.9188.$$

To obtain this result we had only to calculate eigenvalues of matrices 2 by 2 (3 times) and 3 by 3 (once).

4. Continuous time case

In this section we very briefly discuss the continuous time systems of the form

$$\dot{x}(t) = A(r(t))x(t), \qquad (10)$$

where $r(t)$ is a Markov process taking values in a finite set $S = \{1,2,\dots,s\}$, with infinitesimal generator $Q = \left[q_{ij}\right]_{i,j\in S}$, and initial distribution $p = \left[p_j\right]_{j\in S}$, $p_j = \Pr(r(0) = j)$. $A(j)$, $j \in S$ are matrices of size $n \times n$. Solution of (1) with initial condition $x(0) = x_0 \in R^n$ is denoted by $x(k, x_0, p)$. This system is called stable if for all initial conditions $x_0 \in R^n$ and all initial distributions p, we have

$$\lim_{t\to\infty} E\left\|x(t, x_0, p)\right\|^2 = 0.$$

Condition for mean square stability has been given in [8]. It is presented below.

Theorem 3 [8] System (10) is mean square stable if and only if

$$\max\{\mathrm{Re}\, l : l \in s(F_c)\} < 0, \qquad (11)$$

where

$$F_c = \left(P' \otimes I_{n^2}\right) + diag\left[I_n \otimes A_i + A_i \otimes I_n\right]_{i=1,\dots,s} .$$

Using a very similar technique as in the proof of Theorem 2 we may show that (11) holds if

$$\max_{i=1,\dots,s} 2\, l_1\left(A_i + A_i'\right) + l_1\left(Q + Q'\right) < 0 \qquad (12)$$

or if

$$\max \operatorname{Re} l_i(G) < 0, \qquad (13)$$

where

$$G = 2diag\left[a_i\right]_{i=1,\dots,s} + Q + Q'$$

and

$$a_i = \max\left\{\operatorname{Re}\left(l_1 + l_2\right): l_1 \in s\,(A_k),\, l_2 \in s\,(A_l), k, l \in S\right\}.$$

According to Theorem 3 conditions (12) and (13) are sufficient conditions for mean square stability of (10). The advantages of them over (11) are the same as in the discrete time case.

5. Conclusions

In this paper new sufficient condition for mean square stability of discrete and continuous time jump linear system are presented. These conditions are less numerically involved than the known necessary and sufficient conditions.

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