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Exact Analysis of non uniform coupled transmissions lines by numerical solution of Hill's equation

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Abstract: - In this paper, the modeling of two coupled non uniform transmissions lines (NUTLs) is considered. Numerical solution of Hill's equation leads to explicit expressions describing accurately the behavior of these transmission structures.
This technique is applied to the simulation of various profiles and is proven to give good results.

Key-Words: coupled lines, scattering matrix, even and odd modes, Hill equation, non uniform transmission lines.

1 Introduction

The analysis and design of non uniform transmission lines have been extremely investigated in the literature [6], [7]. Various applications of the coupled transmission lines have been of great importance for both microwaves and power engineering.

Besides, non uniform coupled micro strip lines play an important role in both analog and digital microwave integrated circuits. They were successfully used to design, for instance, wideband and tight coupling couplers exploiting non uniform geometries and inhomogeneous media.

To gain some insight into the relationship between the complex geometry and the characteristics of non uniform coupling structures, it is essential to derive exact and explicit expressions describing the main properties of coupled NUTLs in terms of impedance, loss and coupling.

This paper details a robust and universal method, based on Hill's equation solution, to analyze the response of coupled transmission lines excited by lumped sources.

2 Problem Statement

Let us consider an arbitrary non uniform coupled transmission section of length d (fig.1). The TEM mode model of such lossless coupled transmission lines can be described by the following system of partial differential equations:

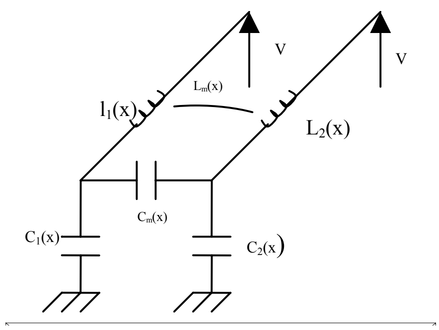

Fig.1 Electric equivalent circuit of two coupled NUTLs.

$$\frac{dV_1(x)}{dx} = -j.\omega.L_1(x).I_1(x) - j.\omega.L_m(x).I_2(x) \quad (1)$$

$$\frac{dI_1(x)}{dx} = -j.\omega.C_1(x).V_1(x) + j.\omega.C_m(x)(V_2(x) - V_1(x)) \quad (2)$$

$$\frac{dV_2(x)}{dx} = -j.\omega.L_m(x).I_1(x) - j.\omega.L_2(x).I_2(x) \quad (3)$$

$$\frac{dI_2(x)}{dx} = -j.\omega.C_m(x).(V_1(x) - V_2(x)) + j.\omega.C_2(x).V_2(x) \quad (4)$$

Where:
$V_i(x)$ (i=1,2) are the voltages at point x along the NUTL.
$I_i(x)$ (i=1,2) are the currents at point x along the NUTL.
$L_i(x)$ (i=1,2) are the self inductances at position x.
$C_i(x)$ (i=1,2) are the self capacitances at position x.
$L_m(x)$ and $C_m(x)$ are the mutual inductance and capacitance, respectively.

All primary parameters are real and per unit length quantities. However, under lossy conditions some of them may have complex values.

An exact solution of the above equations is achieved following the below mentioned procedure:
a. Determination of coupled NUTLs parameters in both even and odd modes.
b. Expression of corresponding scattering matrices in terms of primary parameters.
c. Solution of Hill's equation.

These steps are detailed in the next sections.

3 Scattering Matrices

Coupled transmission lines can be studied by considering even and odd modes approach [2].
Their equivalent 4-ports circuit can be described by a 4x4 S-parameters matrix, S given by:

$$[S]=\begin{bmatrix} s_{11} & s_{21} & s_{31} & s_{41} \\ s_{21} & s_{22} & s_{23} & s_{24} \\ s_{31} & s_{32} & s_{33} & s_{34} \\ s_{41} & s_{42} & s_{43} & s_{44} \end{bmatrix} \quad (5)$$

Due to reciprocity of the passive device as well as symmetry of the structure along the longitudinal plane, this matrix can be written in a more convenient form as follows:

$$[S]=\begin{bmatrix} S_A & S_B \\ S_B & S_A \end{bmatrix} \quad (6)$$

Where: $[S_A]=\begin{bmatrix} S_{11} & S_{21} \\ S_{21} & S_{22} \end{bmatrix}$ and $[S_B]=\begin{bmatrix} S_{31} & S_{41} \\ S_{41} & S_{42} \end{bmatrix}$

Matrices S_A and S_B result from even and odd excitations.

In fact, for even mode, ports 1 and 3 are excited in phase with the same amplitude whereas in odd mode their identical inputs are out of phase.
Under these conditions, and writing reflected waves in terms of incident waves, we obtain:

$$[S_e]=\begin{bmatrix} S_{11e} & S_{21e} \\ S_{21e} & S_{22e} \end{bmatrix}=[S_A]+[S_B]$$

$$[S_o]=\begin{bmatrix} S_{11o} & S_{21o} \\ S_{21o} & S_{22o} \end{bmatrix}=[S_A]-[S_B] \quad (7)$$

Finally, S_A and S_B can be easily derived:

$$[S_A]=\frac{[S_e]+[S_o]}{2}$$

$$[S_B]=\frac{[S_e]-[S_o]}{2} \quad (8)$$

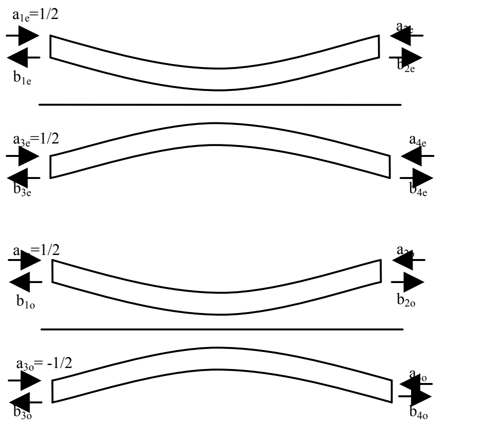

Fig.2 Even and odd mode excitation of NUTLs.

The whole scattering matrix is then written, as follows:

$$[S]=\frac{1}{2}\begin{bmatrix} S_{11e}+S_{11o} & S_{21e}+S_{21e} & S_{11e}-S_{11o} & S_{21e}-S_{21o} \\ S_{21e}+S_{21o} & S_{22e}+S_{22o} & S_{21e}-S_{21o} & S_{22e}-S_{22o} \\ S_{11e}-S_{11o} & S_{21e}-S_{21o} & S_{11e}+S_{11o} & S_{21e}+S_{21o} \\ S_{21o}-S_{21o} & S_{22e}-S_{22o} & S_{21e}+S_{21o} & S_{22e}+S_{22o} \end{bmatrix}$$

Afterwards, given the non uniformity profiles of width w(x) and spacing s(x), one can derive systematically the self and mutual inductances and capacitances namely, $C_m(x)$,$L_m(x)$, $L_0(x)=L_1(x)=L_2(x)$, and $C_0(x)=C_1(x)=C_2(x)$. This leads to the classical expressions of characteristic impedances and propagation constants in both odd and even modes.

$$Z_{ce}=\sqrt{\frac{L_o+L_m}{C_o-C_m}} \quad (9)$$

$$Z_{co}=\sqrt{\frac{L_o-L_m}{C_o+C_m}} \quad (10)$$

$$\gamma_e = j\omega\sqrt{(L_o(x)+L_m(x)).(C_0(x)-C_m(x))} \quad (11)$$

$$\gamma_e = j\omega\sqrt{(L_o(x)+L_m(x)).(C_0(x)-C_m(x))} \quad (12)$$

Substituting these quantities in (1)-(4), and after performing elementary mathematical manipulations the telegraphers equations can be identified to the common form of Hill's equation for both even and odd mode:

$$\frac{\partial^2 U_{e,o}(\xi)}{\partial\xi^2}-\left[\gamma_{e,o}^2(\xi)\left(\frac{d}{\pi}\right)^2+\sqrt{Zc_{e,o}(\xi)\gamma_{e,o}(\xi)}.\frac{\partial^2}{\partial\xi^2}\left(\frac{1}{\sqrt{Zc_{e,o}(\xi)\gamma_{e,o}(\xi)}}\right)\right]U_{e,o}(\xi)=0 \quad (13)$$

where d is the coupled NUTL length.

Solving these equations for both even and odd modes allows the extraction of S_e, S_o and S matrices. This is accomplished by numerical solution of Hill's equation subject of the following section.

4 Solution of Hill's equation

In most propagation problems occurring in inhomogeneous or non uniform structures, the propagation equation can be put, handling basic transformations, in the form of a traditional Hill's equation without a first derivative term [1].

$$\frac{d^2U(\xi)}{d\xi^2} + g(\xi)U(\xi) = 0 \qquad (14)$$

Where:
U(ξ), represents a voltage or one component of electric or magnetic filed.
g(ξ), describes the non uniformity profile.
ξ denotes the longitudinal coordinate.

According to the Floquet theorem [4], the general solution *U(ξ)* is a combination of two linearly independent particular solutions U_1 *(ξ)* and U_2 *(ξ)* written as:

$$U_1(\xi) = e^{\mu 1\xi}.u_1(\xi) \qquad (15)$$

$$U_2(\xi) = e^{\mu 2\xi}.u_2(\xi) \qquad (16)$$

$$U(\xi) = A.U_1(\xi) + B.U_2(\xi) \qquad (17)$$

Where:
A and B are determined by the boundary conditions.
μ_1 and $\mu_{2=}$-μ_1 are the Floquet exponents.
u_1 (ξ), u_2 (ξ) are π-periodical functions expressed by infinite sums of this form:

$$u_1(\xi) = \sum_{-\infty}^{+\infty} C_{1,n}.e^{j2n\xi} \quad \text{and} \quad u_2(\xi) = \sum_{-\infty}^{+\infty} C_{2,n}.e^{j2n\xi} \qquad (18)$$

Where $C_{1,N}$ and $C_{2,N}$ are the coefficients of Fourier series expansion of u_1 (ξ) and u_2 (ξ), respectively.

The above equation (eq.14) can be solved in a systematic fashion by:

a. First expanding g(ξ) in Fourier series :

$$g(\xi) = \sum_{-\infty}^{+\infty} \theta_n e^{j2n\xi} \qquad (19)$$

b. Second, truncating the infinite set of linear and inhomogeneous equations to solve for Floquet's exponents.
c. Finally, writing the general solution in terms of calculated coefficients and exponents as follows:

$$U(\xi) = A.e^{\mu_1\xi}\sum_{-N}^{+N} C_{1,n}e^{j2n\xi} + B.e^{\mu_2\xi}\sum_{-N}^{+N} C_{2,n}e^{j2n\xi} \qquad (20)$$

For that, the g(ξ) expansion combined with a particular solution (U_1 *(ξ)* or U_2 *(ξ)*) are inserted in equation (14) to obtain the resulting infinite set of equations:

$$(\mu + j2n)^2.C_n + \sum_{p=-\infty}^{p=+\infty} \theta_{n-p}.C_p = 0 \quad , \quad n \in Z \qquad (21)$$

It is noteworthy that, Fourier coefficients θ_n of g(ξ) decay rapidly to zero, allowing hence the truncation of this series to a finite and low number of harmonics ensuring sufficient precision.
According to H. Pointcarré and of L. Ince investigations [5], the determinant of the truncated system converges and may be written in the closed-form expression:

$$\Delta(\mu) = \frac{(e^{\pi\mu} - e^{\pi\mu_1})(e^{\pi\mu} - e^{-\pi\mu_1})}{(e^{\pi\mu} - e^{\pi\xi_1})(e^{\pi\mu} - e^{-\pi\xi_1})} \qquad (22)$$

With:

$$\xi_1 = j\sqrt{\theta_0}$$

Floquet exponents, solutions of this equation, are found iteratively by canceling the determinant Δ ($\mu_{1,2}$) = 0, while the $C_{i,,n}$ are assumed different from zero.

5 Validation Results

Several coupled NUTLs having different profiles (linear, parabolic, exponential...) have been analyzed successfully using the abovementioned method.

Their behavior in terms of reflection and transmission coefficients has been observed over a wide frequency band.
The effects of geometrical shapes as well as dielectric permittivity profiles have been assessed.

Results obtained for two coupled lines with a linear profile of width w(x) and exponential variation of spacing s(x) (Figure 3) are depicted in Figures 4,5.

6 Conclusion

A universal and systematic approach has been proposed to analyze coupled non uniform transmission lines.
After evaluating primary parameters, based on geometry and media considerations, characteristic impedances in odd and even modes are determined.

The solution of Hill's equation is achieved using an efficient iterative method based on Floquet exponents determination.
Once voltages and currents are defined over each point x along the transmission structure, S-parameters and other pertinent features can be easily derived.
To illustrate the accuracy, efficiency and versatility of the proposed method, various representative structures have been simulated and showed good agreement with measurements.

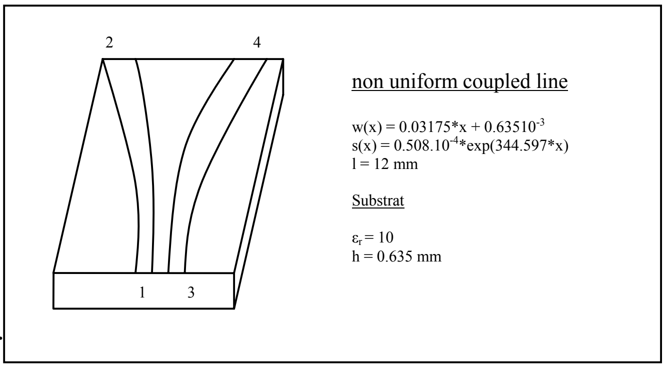

Fig.3 Two non uniform coupled lines.

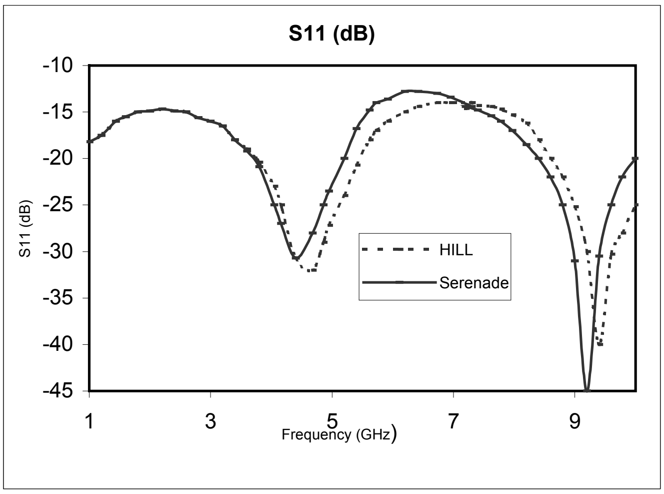

Fig.4 S_{11} of two non uniform coupled lines.

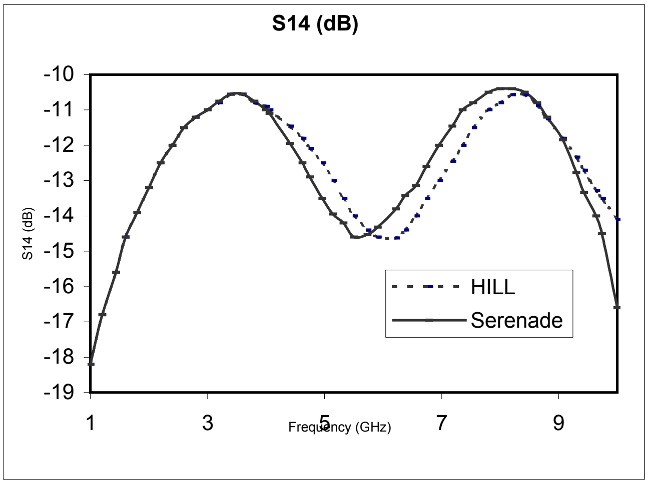

Fig.5 S_{14} of two non uniform coupled lines.

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