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# Distance Spectrum Analysis of Third Generation Turbo Codes

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*Abstract:* Turbo Codes are a class of powerful error correction codes that were introduced in 1993 by a group of researchers from France, which has the performance near the limit of Claude Shannon. After the introduction of turbo codes it has given raise a tremendous research work related to the new coding theory. This paper addresses the performance of Turbo codes by examining the codes' distance spectrum. It is well known that error floor occurs in the performance curve of turbo codes at moderate to high signal-to-noise ratio. The cause of error floor is due to the relatively low free distance of the codewords. Several techniques were proposed by researchers to lower the error floor. These techniques are assessed in this paper. To determine the free distance several algorithms were developed by different researchers. In this paper we used one of the recent algorithm to evaluate the distance spectrum of Turbo codes. We concentrate our analysis to measure and explain the distance spectrum of UMTS (Universal Mobile Telecommunication System), cdma2000 and CCSDS (Consultative Committee for Space Data Systems) standards Turbo Codes. It is shown that the distance spectrum depends on the code rate, interleaver size and the interleaver type.This distance spectrum of turbo codes can be used to estimate its performance at medium to higher SNR (signal to noise ratio). From our analysis we find out that the distance spectrum is one of the elementary issues using which one can find the optimum architecture of Turbo codes for specific application.

*Keywords:* Turbo codes, Wireless communication, Distance spectrum, Error floor, UMTS, cdma2000, CCSDS.

## 1 Introduction

Berrou et. al. first discovered the turbo codes and reported its outstanding performance in [1]. Initially greeted with some skepticism, the original results were independently reproduced by several researchers [2]-[5]. After the confirmation of the result by Turbo codes, researchers try to focus their research on the understanding of excellent performance of the Turbo codes [6]-[8].

This paper will define and evaluate the upper bound to the average performance of the decoder for a Parallel Concatenated Convolutional codes (PCCC) [9]. The upper bound of Turbo codes can be expressed by the following equation.

$$P_b \le \sum_{d=d_{free}}^{2(v+N)} \frac{N_d \tilde{w}_d}{N} Q\left(\sqrt{d\frac{2RE_b}{N_0}}\right) \quad \text{............ ...(1)}$$

Where $P_b$ is error probability, $v$ is the memory of convolutional code, $d$ is the Hamming distance, $N$ is the interleaver size, $N_d$ is the multiplicity of weight-$d$ codeword, $\tilde{w}_d$ is the average weight of the information sequences causing weight-$d$ codeword, $R$ is the code rate, $E_b$ is the signal energy and $N_0$ is the noise spectral ratio.

We can define the free distance term, which is the minimum Hamming distance between the codeword and all zero codeword. It is found that the free distance dominates on the BER performance of Turbo codes at medium to high SNR. It is found that at medium to high SNR error floor occurs in the BER performance. The error floor is the flattening part of the performance curve, for moderate to high SNR. There was a question, what causes the "error floor" [10]? Many researchers prove that the error floor is mainly due to the free distance of turbo codes [11]. In this region, its free distance, $d_{\text{free}}$ and its multiplicities, dominates the performance of any binary code. As mentioned in [12], some concatenated codes with interleavers may have very low free distances, even when large interleaver lengths *N* are used. This causes their BER curves to flatten according to the "error floor" imposed by $d_{free}$, after the "water fall" part of the curve. This behavior is not expected for applications requiring very low BERs, e.g., between $10^{-6}$ and $10^{-10}$. So to assess the performance of Turbo codes, we have to measure the free distance of Turbo codes of a definite structure. We will explain the algorithm, developed by the R. Garello et. al. in [13] to find the distance spectrum of Turbo codes. The algorithm is improved by E. Rosnes et. al. in [14]. In this paper this algorithm is implemented to measure the distance spectrum of third generation standard turbo code, i.e., UMTS, cdma2000 and CCSDS turbo codes. Then we show the dependence of distance spectrum on code rate, interleaver size and interleaver type.

In section 2 the structure of classical turbo code and its performance is explained. Performance bound of turbo code is explained in section 3. The method of lowering the error floor and the algorithm to measure the distance spectrum of turbo codes are analyzed in section 4 and 5. Then by implementing the algorithm we have measured the distance spectrum of UMTS, cdma2000 and CCSDS turbo codes and show how the code rate and interleaver size influence the performance of turbo code in section 6 and finally we conclude in section7.

## 2 Turbo Codes

In order to explain the performance of Turbo codes in terms of the free distance and the distance spectrum, we examine the codeword structure of Turbo codes in detail. We use specific example of Turbo codes to elucidate the key structural properties. The encoder is shown in Fig.1. This encoder is rate 1/3 and 4 state trellis code. The constituent encoder is shown in Fig.1 and Fig. 2.

The simulated performance of a rate 1/2 Turbo code with the same parameters as in [15] is reproduced in Fig.3. The 'ab' portion of the performance curve is "waterfall" region and 'bc' portion is the "error floor" region. In this paper we will study the error floor region, why it occurs, its measurement in terms of free distance and distance spectrum and how this performance can be improved.

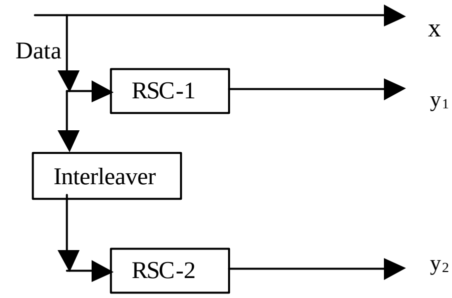


Fig.1. Basic Turbo Encoder of rate 1/3 ($x_1$ is systematic bit, $y_1$ and $y_2$ are parity bits)

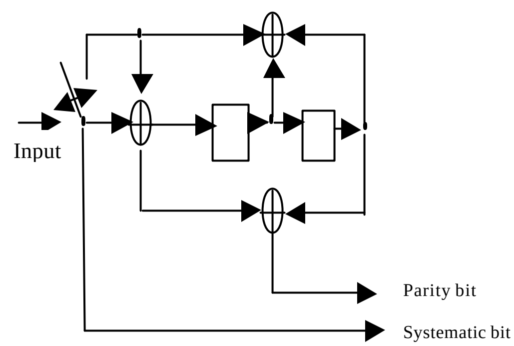


Fig. 2. Turbo RSC Encoder

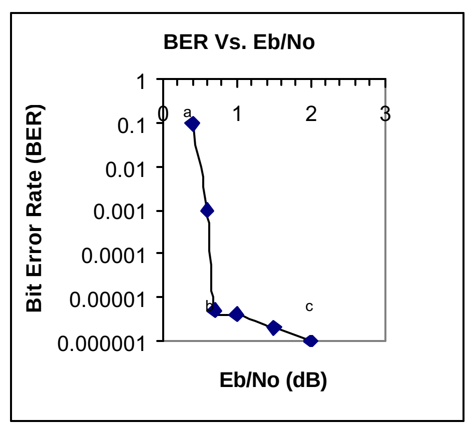


Fig. 3: BER performance of the classical turbo codes.

## 3 Performance Bound of Turbo codes

The bit error rate (BER) performance of a convolutional code with maximum-likelihood (ML) decoding on an additive white Gaussian noise

(AWGN) channel can be upper-bounded using a union bound technique by [16]

$$P_b \le \sum_{i=1}^{2(v+N)} \frac{w_i}{N} Q\left(\sqrt{d_i \frac{2RE_b}{N_0}}\right) \quad (2)$$

Where, $w_i$ is information weight and $d_i$ is total Hamming weight of the $i^{th}$ codeword. Let us define the average information weight per codeword as

$$\tilde{w}_d = \frac{W_d}{N_d}$$

Here $W_d$ is the total information weight of all code words of weight $d$ and $N_d$ is the multiplicity of code words of weight $d$. So the Eq. (2) becomes:

$$P_b \le \sum_{d=d_{free}}^{2(v+N)} \frac{N_d \tilde{w}_d}{N} Q\left(\sqrt{d \frac{2RE_b}{N_0}}\right) \quad (3)$$

Eq. (3) is the upper bound for the convolutional code. The performance of a Turbo code with maximum-likelihood decoding can also be bounded using the union bound of Eq. (3). For moderate and high signal-to-noise ratios, it is well known that the free-distance term in the union bound on the bit error rate performance dominates the bound. Thus for a turbo codes the asymptotic performance approaches:

$$P_b \approx \frac{\tilde{w}_{free} N_{free}}{N} Q\left(\sqrt{d_{free} \frac{2RE_b}{N_0}}\right) \quad (4)$$

Where $N_{free}$ is the multiplicity of free-distance codewords and $\tilde{w}_{free}$ is the average weight of the information sequences causing free-distance codewords. By using algorithm for finding the free distance and plugging the values in Eq. (4), the free distance asymptotes graph can be generated. For the turbo codes in [15], i.e., (37, 21, 65536) code was found to have $N_{free}$=3 paths, $d_{free}$ = 6. For this particular Turbo code, the free distance asymptote is given by

$$P_{free} = \frac{3 \times 2}{65536} Q\left(\sqrt{6 \frac{E_b}{N_0}}\right) \quad (5)$$

By plotting $P_{free}$ vs. $\frac{E_b}{N_0}$ we get the asymptotic curve in Fig. 3.

For this code the effective multiplicity is

$$\frac{N_{free}}{N} = \frac{3}{65536}$$

The free-distance asymptotes are shown in Fig. 4 and the simulation result is shown in Fig. 3. From these figures it can be clearly seen that the simulation result do in fact approach the free-distance asymptote for moderate and high SNR. Since the slope of the asymptote is essentially determined by the free distance of the code, it can be concluded that the error-floor observed in the Turbo codes performance is due to the fact that they have a relatively small free distance and consequently a relatively flat free-distance asymptote.

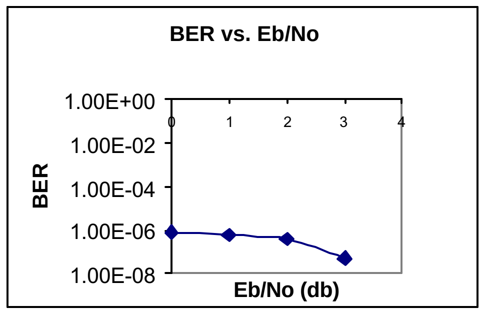


Fig. 4: Upper bound of the classical turbo codes.

## 4 Lowering the Error Floor of Turbo Codes

From [12] we found that increasing the length of the interleaver while preserving the free distance and the multiplicity, will lower the asymptote without changing its slope by reducing the effective multiplicity. In this case the performance curve of Turbo codes does not flatten out until higher SNR's and lower BER's are reached. If the size of the interleaver is fixed, then increasing the free can modify the error floor distance of the code while preserving the multiplicity. This has the effect of changing the slope of the free-distance asymptote. That is, increasing the free distance increases the slope of the asymptote and decreasing the free distance, decreases the slope of the asymptote. It also shown in [7] and [9] that for a fixed interleaver size, choosing the feedback polynomial to be a primitive polynomial result in an increased free distance and thus a steeper asymptote.

Another way to improve the error floor is that [17], we have to identify the information bit positions affected by low-distance error event, which are few in number due to the sparseness of the spectrum. A modified encoder inserts dummy bits in these positions, resulting in a lower and steeper error floor the bit-error-rate performance curve. For sufficiently large interleaver size, the only cost is a very slight reduction in the code rate.

# 5 Distance Spectrum Measurement of Turbo Codes

Different efforts were taken to measure the distance spectrum of turbo codes as in [13], [18]-[19]. In this paper we use the recent algorithm as described in [13] to evaluate the distance spectrum of different turbo codes as standard of UMTS, cdma2000 and CCSDS. First we describe the algorithm of [13] in different notations.

## 5.1 Definition

A constrained set $F$ is defined as $\left\{\left(p_i, u_{p_i}\right) : u_{p_i} \in \{0, 1\} \ \forall p_i \in F_p\right\}$

$F_p$ is defined as $F_p \subseteq \{0, 1, \ldots\ldots N-1\}$

$U^{(F)}$ be the set of length-$N$ vectors defined as

$$\left\{\begin{matrix} u = (u_0, \ldots, u_{N-1}) : u_j = u \text{ if } (j,u) \in F,\ u_j \in \{0,1\} \\ \text{if } j \notin F_p \end{matrix}\right\}$$

length $l = l(F)$ be the number of constraints.

Turbo interleaver acting on $F$, we obtain a new constraint set $pF = \left\{\left(p(p_i), u_{p_i}\right)\right\}$.

$C^{(F)}$ be the subset of the turbo code and we get it by encoding the input vectors in $U^{(F)}$.

$w(F)$ be the minimum hamming weight of $C^{(F)}$.

## 5.2 Algorithm

Add an empty constraint set $F$ to a previously empty list L of constraint sets.

(*) If $L$ is empty,

terminate the process.

Otherwise,

choose and take out a constraint set $F$ from $L$.

(**) If $w(F) \le t$, then

If the length $l$ of $F$ is equal to $N$ then:

The single vector in $U^F$ produces a low-weight word. Make a record of it.

Otherwise,

construct two new constraint sets:

$F' = F \cup \{l, 0\}$, and

$F'' = F \cup \{l, 1\}$

Add $F'$ and $F''$ to L.

Proceed from (*)

Finally the turbo code free distance, code multiplicity and information multiplicity can be evaluated. In [14] some improvements of the stated algorithm are proposed.

# 6 Distance Spectrum of Turbo Codes of Different Standards

The preceding algorithm can be implemented to compute all the terms of distance spectrum together with their multiplicities. Applying the algorithm we compute the distance spectrum of UMTS, cdma2000 and CCSDS turbo codes.

## 6.1 UMTS Turbo Codes

UMTS is third generation partnership project (3GPP) standard. Its encoder has two recursive systematic convolutional encoder. Each convolutional encoder is 8-state and 1/2 rate and the rate of turbo codes is 1/3. A block interleaver with length $N$ is used. Its range is in between 41 to 5114. The algorithm of the interleaver is described in [20].

The distance spectrum of UMTS turbo codes is reported in table 3. Here $d_{free}$ is the free distance, $N_{free}$ is its multiplicities and $w_{free}$ is its information multiplicities.

Table 3

| Distance spectrum | $d_{free}$ | $N_{free}$ | $w_{free}$ |
|---|---|---|---|
| $N$ = 40 | 13 | 3 | 9 |
| $N$ = 200 | 20 | 1 | 4 |
| $N$ = 320 | 24 | 1 | 4 |

From Table 3, it is found that the free distance, $d_{free}$ is increased with the increase of the interleaver size. At the same time the codeword multiplicities, $N_{free}$ and the information multiplicities, $w_{free}$ is decreased with the increase of the interleaver length $N$. So the performance of the Turbo codes improves with the increase of interleaver size, which was predicted.

### 6.3 cdma2000 Turbo Codes

cdma2000 is third generation partnership project-2 (3GPP2) standard. Its encoder has two recursive systematic convolutional encoder. Each convolutional encoder is 8-state encoder. The rate of turbo codes is 1/2, 1/3, 1/4, 1/5. The transfer function of turbo code can be expressed as

$$G(D) = \left[1 + \frac{n_0(D)}{d(D)} + \frac{n_1(D)}{d(D)}\right]$$

Where, $n_0(D) = 1 + D + D^3$, $n_1(D) = 1 + D + D^2 + D^3$ and $d(D) = 1 + D^2 + D^3$.

Table 4

| cdma2000 turbo codes | $d_{free}$ | $N_{free}$ | $w_{free}$ |
|---|---|---|---|
| rate = 1/2 | 10 | 1 | 3 |
| rate = 1/3 | 21 | 5 | 15 |
| rate = 1/4 | 30 | 2 | 6 |

The puncturing pattern and the interleaving technique can be found in [21]. The distance spectrum is reported in table 4. This table demonstrate that if the code rate is decreased the free distance, $d_{free}$ is increased.

### 6.4 CCSDS Turbo Code

In CCSDS (Consultative Committee for Space Data Systems) standards, the old channel coding standard has been updated to include turbo codes [22]. The encoder is two equal binary systematic recursive convolutional encoders with rate 1/4 and 16 state terminated in 4 steps. The block interleaver length is 1784, 3568, 7163 or 8920. The algorithm of the interleaver is described in [22]. The code rate is 1/2, 1/3, 1/4 and 1/6. By applying the algorithm to measure the distance spectrum we get the spectrum, which is reported, in Table 5 and Fig. 5.

Table 5

| | Interleaver Size | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | N = 1784 | | | | N = 3568 | | | |
| Rate | 1/2 | 1/3 | 1/4 | 1/6 | 1/2 | 1/3 | 1/4 | 1/6 |
| $d_{free}$ | 17 | 32 | 42 | 70 | 20 | 40 | 56 | 93 |
| $N_{free}$ | 2 | 1 | 1 | 1 | 1 | 3 | 2 | 2 |
| $w_{free}$ | 6 | 2 | 2 | 2 | 1 | 9 | 6 | 6 |

From the table and figure, it is found that for a fixed interleaver size, if the code rate is decreased the free distance is increased, that the performance is improved. It is also shown in the figure that, if the interleaver size is increased, for same code rate, the free distance is also increased.

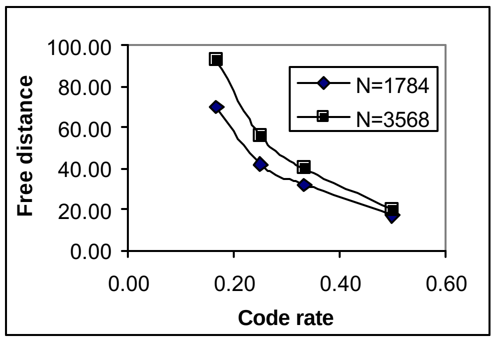


Fig.5 Free distance vs. code rate for different interleaver size.

## 7 Conclusion

This paper has analyzed the distance properties of turbo codes and its relation with its error floor region of performance curve. Techniques that are used to improve the error floor region, i.e., to lower the error floor or to negatively increase the slope of the error floor, has discussed. There are number of algorithms that are used to evaluate the distance spectrum. Among those we have explained the algorithm proposed by R. Garrelo et. al., using which we can measure the distance spectrum of turbo codes. We have implemented the algorithm to measure the distance properties, i.e., the free distance, its code multiplicities and the corresponding information multiplicities of turbo codes. We have measured the distance spectrum of Third Generation standard (mainly UMTS and cdma2000) turbo codes and the CCSDS turbo codes. It is shown that if we decrease the code rate, the free distance is increased. It is also shown that if we increase the interleave size, the free distance is increased. The distance spectrum of third generation and CCSDS turbo codes is needed to estimate its performance at low BER ($10^{-6}$ to $10^{-9}$) and medium to high bit energy to noise spectral ratio.

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