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# Matrix Norms and their Sensitivity to Noise a Computational Study

**M. O. Abdalla***
Department of Mechanical Engineering
University of Jordan
Amman-Jordan
admin@mechatronix.us

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## Abstract

Most of the objective (cost) functions in optimization techniques utilize norms especially when dealing with signals, vectors, or matrices. In this work, three norms were studied, namely matrix One norm, Infinity norm, and Frobenius (Euclidean or Two) norm. The effect of noise on these matrix norms was studied with the aid of a generalized eigen equation. Basic analysis of the effect of noise on matrix norms is provided, which is also complimented with a computer simulated results. It turns out that the Frobenius norm is the least sensitive norm to noise.

Key-Words: Optimization, Noise, 1-norm, Infinity-norm, Frobenous-norm, Eigenequation.

## 1 Introduction

In optimal control theory many of the goals a controller needs to achieve maybe expressed interms of the size of various signals, for an example in a tracking problem the error signals should be made small, while the actuator signals should be bounded. Signal sizes are best captured using norms, because of their nice geometric properties, which is mainly expressed in the framework of a vector space [1]. Matrices, which may be thought of as vectors in a higher dimensional spaces, sizes are of concern in optimization and cost function minimization. Norms of matrices maybe thought of as a generalizations of Eulidean length. Also different acceptable norms maybe more or less convenient in various situations [2].

Adetailed list of matrix norms applications in engineering was compiled by G. Belitskii and Y. Lyubich [3]. They had showed the importance of matrix norms in different engineering problems. For example in the control theory performance indices in the time domain hinge on norms, good examples maybe found in $\mathcal{H}_\infty$ , LPV, and LMI designs [4][5].

## 2 Norms Overview[2][6]

It is interesting to know that a significant portion of linear algebra is infact geometric in nature, which stemmed from the need of generalizing higher dimensional spaces. Actually, questions of size and proximity in a two dimensional or three dimensional vector spaces usually refer to Euclidean distance, but

*Corresponding author. E-mail: admin@mechatronix.us. Tel: (9626) 5355000, ext. 2768, and Fax: (9626) 5355588.

what we can say about vectors in infinite-dimensional spaces! or what about the size of matrices (vectors in a higher dimensional space)! One way to quantify these concepts is by using norms, which may measure the size and proximity of vectors and matrices. Basically, norms are defined for signal functions, vectors, and matrices. Nevertheless, in this work we concentrate on norms of matrices.
Matrix norms definition and their properties are provided as an overview.

**Definition 1** *A matrix norm is a function* $\|*\|$ *from the set of all complex matrices (finite orders) into* $\mathcal{R}$ *that satisfies the following properties*

$$\begin{array}{ll} \|A\| \geq 0 \text{ and } \|A\| = 0 \Leftrightarrow A = 0 & \\ \|\alpha A\| = |\alpha| \, \|A\| & \forall \alpha \in \mathcal{R} \\ \|A+B\| \leq \|A\| + \|B\| & \\ \|AB\| \leq \|A\| \, \|B\| & A, B \in \mathcal{R}^{n\times n} \end{array}$$

The most used matrix norms in optimization problems are the 1-norm, 2-norm, Frobenous norm, and the $\infty$-norm. Please note that older texts refer to the Frobenous norm as the *Hilbert-Schmidt norm or Schur norm.* A summary of the definitions of these norms is provided here for reference.

**Definition 2** *Frobenous Matrix Norm of* $A \in \mathcal{R}^{m\times n}$ *is defined by the equations*

$$\|A\|_F^2 = \sum_{i,j} |a_{ij}|^2 = trace(A^T A)$$

**Definition 3** *Matrix* 2*-norm* $A \in \mathcal{R}^{m\times n}$ *induced by the Euclidean vector norm is*

$$\|A\|_2 = \max_{\|x\|_2 = 1} \|Ax\|_2 = \sqrt{\lambda_{\max}} \qquad \ni \ |A| \neq 0$$

**Definition 4** *Matrix* 1*-norm and Matrix* $\infty$*-norm induced by a vector* 1*-norm and Matrix* $\infty$*-norm are as follows*

$$\begin{aligned} \|A\|_1 &= \max_{\|x\|_1=1} \|Ax\|_1 = \max_j \sum_i |a_{ij}| \\ \|A\|_\infty &= \max_{\|x\|_\infty=1} \|Ax\|_\infty = \max_i \sum_j |a_{ij}| \end{aligned}$$

# 3 Problem Formulation

To illustrate the effect of noise on matrix norms we will consider the generalized eigen equation [7] as a case study. The eigen equation is widely used in mechanical structures, so lets consider an $n$ dimensional (DOF) Finite Element (FE) model of a mechanical structure, which maybe given by the following equation of motion,

$$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = f(t) \qquad (1)$$

where $M$, $C$, and $K \in \mathcal{R}^{n\times n}$ are the analytical mass, damping, and stiffness matrices respectively, and $q \in \mathcal{R}^{n\times 1}$ is the displacements vector. Hence, the corresponding matrix eigen equation maybe given as

$$KV - MV\Omega^2 = 0 \qquad (2)$$

where $\Omega \in \mathcal{R}^{n\times n}$ is a diagonal matrix of the eigen frequencies (eigenvalues) for the system and $V \in \mathcal{R}^{n\times n}$ is the corresponding eigen mode shapes (eigen vectors or modal matrix). The stiffness matrix $K$ maybe written in a parametrized matrix form as follows

$$K = \sum_{i=1}^{n} P_i K_i \qquad (3)$$

where $P_i \in \mathcal{R}$ are set of scalers (weights) and $K_i \in \mathcal{R}^{n\times n}$ are the individual stiffness matrices. Now, the optimization problem is basically to estimate the weight scalers such that it satisfies the aforementioned eigen equation inspite of the presence of noise in the eigen

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vectors. It was evident from Modal Analysis [8] that we can measure frequencies (eigen values) accurately but the eigen vectors (mode shapes) tend to be more susceptible to noise contamination. To illustrate the matrix norms sensitivity to noise consider the following parameter optimization problem

$$\min_{P_i} \left\| KV - MV\Omega^2 \right\|_x \tag{4}$$

where $x$ is a suffix that designates the considered norms (i.e. 1 one, $F$ Frobenous, and $\infty$ infinity norms).

## 4 Noise Effect on Norms

The investigations of the aformensioned eigenproblem using different norms and with mode shapes noise contaminations had revealed the folowing results.

**Theorem 5** *The effect of noise on the Frobenous norm by solving the eigen equation is quadratic and it preserves an extrema.*

**Proof.** Consider the optimization problem depicted by equation (4).

Let $\tilde{K} = \sum_{i=2}^{n} P_i K_i \Rightarrow K = P_1 K_1 + \tilde{K}$

and let $V = \tilde{V} + \delta$ where $\delta$ is the noise vector

$\left\| \left(P_1 K_1 + \tilde{K}\right) V - MV\Omega^2 \right\|_F^2$

$\left\| (P_1 K_1 V + KV - MV\Omega^2 \right\|_F^2$

Let $B = KV - MV\Omega^2$

$\left\| P_1 K_1 V + B \right\|_F^2 \leq$

$\langle P_1 K_1 V, P_1 K_1 V \rangle + 2 \langle P_1 K_1 V, B \rangle + \langle B, B \rangle$

$= \| P_1 K_1 V \|_F + 2 \langle P_1 K_1 V, B \rangle + \| B \|_F$

$\leq \left( \| P_1 K_1 V \|_F + \| B \|_F \right)^2$

$\leq \left( \left\| P_1 K_1 \tilde{V} + P_1 K_1 \delta \right\|_F + \| B \|_F \right)^2$

$\leq \left( \left\| P_1 K_1 \tilde{V} \right\|_F + \| P_1 K_1 \delta \|_F + \| B \|_F \right)^2$ ■

**Theorem 6** *The effect of noise on the* 1-*norm and* $\infty$-*norm by solving the eigen equation is linear and it smears out any extrema.*

**Proof.** For the One and the Infinity norms, define

$A_o = \left(P_1 K_1 + \tilde{K}\right) \left(\tilde{V} + \delta\right) - M \left(\tilde{V} + \delta\right) \Omega^2$

$= P_1 K_1 \tilde{V} + P_1 K_1 \delta + \tilde{K}\tilde{V} + \tilde{K}\delta - M\tilde{V}\Omega^2 - M\delta\Omega^2$

Taking

$P_1 K_1 \delta = [P_1 K_1 \delta]_{ij} = P_1 \sum_{k=1}^{n} K_{1ik} \delta_{kj}$

$\tilde{K}\delta = \left[\tilde{K}\delta\right]_{ij} = \sum_{k=1}^{n} \tilde{K}_{ik} \delta_{kj}$

$M\delta\Omega^2 = \left[M \left(\delta\Omega^2\right)\right]_{ij} = \sum_{k=1}^{n} [M\delta]_{ik} \, \omega_{kj}^2$

$\tilde{A} = P_1 K_1 \tilde{V} + \tilde{K}\tilde{V} - M\tilde{V}\Omega^2$

$A_o = \tilde{A} + P_1 K_1 \delta + \tilde{K}\delta - M\delta\Omega^2$

and let

$A_o = \tilde{A} + A$

$[A]_{ij} = P_1 \sum_{k=1}^{n} K_{1ik} \delta_{kj} + \sum_{k=1}^{n} \tilde{K}_{ik} \delta_{kj} - \sum_{k=1}^{n} [M\delta]_{ik} \, \omega_{kj}^2$

$= \sum_{k=1}^{n} P_1 K_{1ik} \delta_{kj} + \tilde{K}_{ik} \delta_{kj} - [M\delta]_{ik} \, \omega_{kj}^2$

But for the 1-norm

$\| A_o \|_1 = \max_j \sum_{i=1}^{n} |a_{ij}|$

$= \max_j \sum_{i=1}^{n} \left| [A]_{ij} + \left[\tilde{A}\right]_{ij} \right|$

and for the $\infty$-norm

$\| A_o \|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$

$= \max_i \sum_{j=1}^{n} \left| [A]_{ij} + \left[\tilde{A}\right]_{ij} \right|$ ■

The above theorems simply state the fact that certain norms are good for certain applications, and further analysis should be taken into account the noise effect. A numerical example is provided to shed more light on this important issue.

# 5 Numerical Example

In order to demonstrate the analytical results in the previous section we will consider a numerical example. In this example it is needed to minimize the eigen equation norm over the value of $P_1$. A simple searching technique reveals the correct value as depicted in Figure 1. In the noise free case all the norms simultaneously revealed the correct answer.

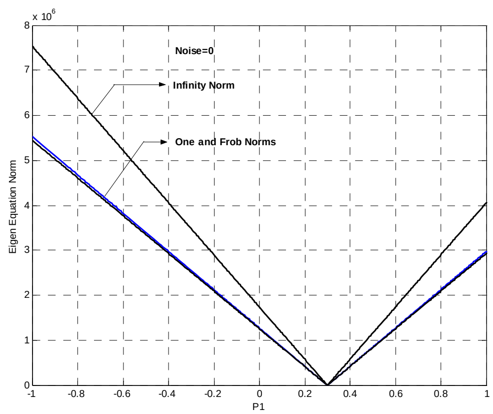


Fig. 1, Noise free case

Adding 5% noise to the eigenvectors (mode shapes in structures lingo) and assuming the eigenvalues (frequencies) are noise free (i.e. the assumption is valid because we can always measure frequencies accurately compared with the mode shapes). Figure 2, illustrates the results, which shows clearly that the one-norm and the infinity-norm are affected linearly by the noise. These norms have smeared out the minimum point, while the Frobenous norm showed a quadratic behavior. The Frobenous norm minimal point is still close to the actual value inspite of the noise.

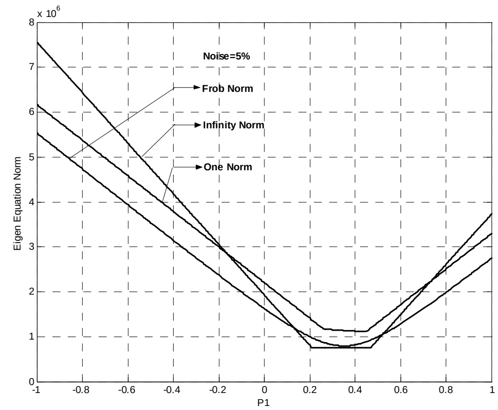


Fig. 2, smear out effect, 5% noise

Finally, a 10% noise is added to the eigenvectors and the results are summarized in Figure 3. Here the Frobenous norm is still pointing out to the minimal point neighborhood, while the other two norms completely smearing out the minimal point. It is evident that an optimization problem with a Frobenous norm objective function will suffer less under noise from the other two considered norms

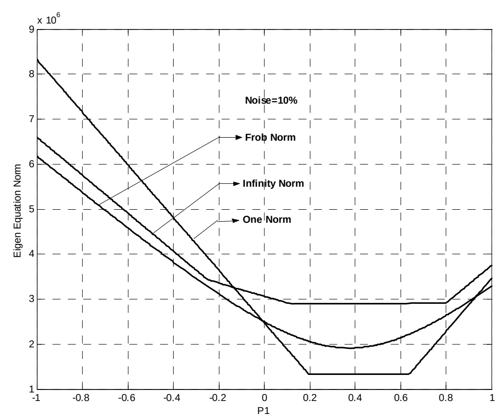


Fig.3, smear out effect, 10% noise

# 6 Conclusions

A short study of matrix norm's and their sensitivity to noise was presented. The main three types of matrix norms were examined

namely: one-norm, infinity-norm, and Frobenous norm. Noise effect on norm based optimization problems was illustrated using the generalized Eigenvalue problem. A sketch of Mathematical proofs for the problems were presented, which were also supported by numerical results. The Analytical and Numerical analysis were in agreement and they have revealed that the Frobenous norm was the least sensitive norm to noise.
Further investigations are needed to be carried out over the sensitivity of the 2-norm (Euclidean norm) to noise. This norm is interesting in particular because of its direct relation to matrix eigen values.

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