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New Bounds on Representation Errors in Signal Processing Systems

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Abstract: - Most signal processing systems are based on discrete-time signals although the origin of many sources of information is analog. In this work we consider the task of signal representation by a set of basis functions. Presently, without prior knowledge of the signal beyond its samples, no bound on the potential representation error is available. The question raised in this paper is to what extent the sampling process keeps algebraic relations, such as inner product, intact. By interpreting the sampling process as a linear bounded operator, an upper bound on the representation error is derived and demonstrated. Based on our theorems, one can then determine the maximum representation error induced by the sampling process. We further propose a new approximation scheme for the calculation of the inner product, which is optimal in the sense of minimizing the maximum representation error. Our results are applicable to signal processing systems where analog signals are represented by their sampled versions.

Key-Words: - Sampling, Inner-Product, Sobolev space, Image Processing

1. Introduction

Signal processing applications are concerned mainly with digital data, although the origin of many sources of information is analog. Such signals include for example speech and audio, optical signals, radar and sonar, biomedical signals and images. Representing a continuous-time (analog) signal by its samples has been widely used since Nyquist formulated the sampling theorem. It is also well known that applying this representation scheme to non bandlimited signals introduces approximation errors: the signal does not belong to $Span\{sinc(\cdot)\}$ and the samples do not correspond to its orthogonal projection. Indeed, according to the sampling theorem, these errors become smaller as the sampling interval shortens ([13]). However, there are cases in which achieving a low approximation error requires high, unrealizable, sampling rates. For this reason, mainly, alternative basis functions such as Gabor functions, wavelets, Hermite functions, Legendre functions, Laguerre functions and the like are often used instead ([4], [5],[6],[9],[10],[11], [12]).

Finding representation coefficients for these alternative bases involves inner-product calculations within the analog domain, rather than simply consider the sampled version of the signal itself as in the bandlimited case. When the signal is not given analytically, this in turn is somewhat difficult to implement. It is even impossible to perform in cases where the signal is already given by its samples only. To overcome this difficulty, it is acceptable to approximate the original inner product by the discrete-time approximation:

$$\langle f,\varphi\rangle \cong T\cdot\sum_n f(nT)\cdot\varphi(nT), \tag{1}$$

where $f(t)$ is the original signal and $\varphi(t)$ is a known (basis) function. Here too, relying on the sampling theorem, the error of this approximation scheme becomes smaller as the sampling interval shortens. However, by having no prior knowledge of the original signal $f(t)$, except for its samples, no bounds on the resulted approximation error are presently available.

Keeping the basic approximation scheme, abovementioned, the question raised and considered in this work is whether the sampling process keeps algebraic relations, shared within the analog domain, intact. We consider the operation widely used in vector representation, the inner product, and propose a new discrete approximation scheme for this calculation, allowing an optimal approach to this widely used approximation.

2. The Problem

We address the following problem (Fig. 1): given a function $\varphi(t)\in L_2$, how can one optimally approximate the inner product of $\langle f,\varphi\rangle$, by having only the samples of $f(t)$? Furthermore, if calculated this way, what is the approximation error?

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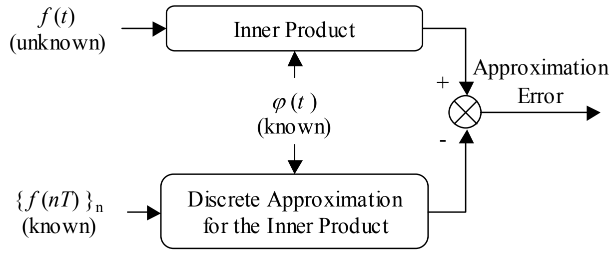


Figure 1: Statement of the problem - given a function $\varphi(t)$, how can one optimally approximate the inner product of $\langle f,\varphi\rangle$, by having only the samples of $f(t)$?

3. Sampling as a Linear Operator

Blu and Unser [4] have shown that sampling a Sobolev function [2] of order one, i.e. $f(t), f'(t) \in L_2$, yields a finite energy sequence. The importance of this result resides in the fact that the sampling process can now be considered as a linearly bounded operator [7] acting on Sobolev functions of an arbitrary order, n, to obtain an l_2 sequence. These functions are dense in L_2, therefore, restricting our analysis to such functions still maintains generalization of the results.

Lemma 1: The sampling operator S_T is given by,

$$\begin{aligned} &S_T : W_2^n \to l_2 \\ &S_T f = \sum_n \langle f(t), u(t-nT)\rangle_{W_2} \cdot e_n \end{aligned}, \quad (2)$$

where $\{e_n\}$ is the standard basis of l_2, and $u(t)$ is the inverse Fourier transform of ,

$$U(\omega) = \frac{1}{1+\omega^2+\omega^4+\cdots+\omega^{2n}}. \quad (3)$$

Lemma 2: The adjoint operator of S_T, namely S_T*, is given by,

$$\begin{aligned} &S_T^* : l_2 \to W_2^n \\ &(S_T^* b)(t) = \sum_n b[n]\cdot u(t-nT) \end{aligned}, \quad (4)$$

where $u(t)$ is given in Lemma 1.

Lemma 3: Let $\varphi(t) \in L_2$ be a known function, and let $b[n] \in l_2$ be a known sequence. Then (Fig. 2), for any $f(t) \in W_2$:

$$\langle f,\varphi\rangle_{L_2} - \langle S_T f, b\rangle_{l_2} = \left\langle f, \varphi^* - S_T^* b\right\rangle_{W_2}, \quad (5)$$

where S_T is the uniform sampling operator given in Lemma 1, S_T* its adjoint as given in Lemma 2. $\varphi^*(t) = \varphi(t) * u(t)$, and $u(t)$ is given in Lemma 1 as well.

The proofs of the above lemmas are given in [8].

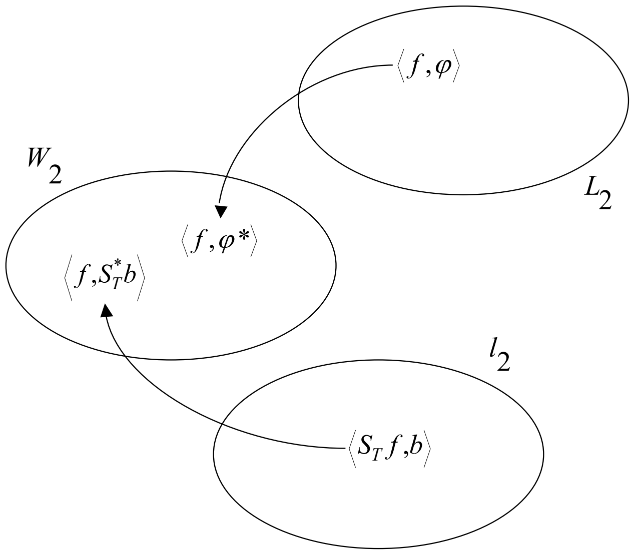


Figure 2: Intertwining relations of L_2, l_2 and W_2 Inner Products. φ is a known function of L_2 and b is a known sequence of l_2. f is an arbitrary function of W_2 to be uniformly sampled. The inner product of L_2 has a corresponding representation in W_2, and the same holds for the l_2 inner product.

4. Sampling Effects on the Inner Product

Theorem 1: Let $\varphi(t) \in W_2$ be a known function. Given a sampling interval T, the following relation holds for any Sobolev function $f(t) \in W_2^n$:

$$\left|\langle f,\varphi\rangle_{L_2} - T\langle S_T f, S_T\varphi\rangle_{l_2}\right| \le B\cdot\|f\|_{L_2}, \quad (6)$$

where S_T is the uniform sampling operator with interval T, B is given by,

$$B = \frac{\left\|\varphi(t)*u(t) - T\sum_n \varphi(nT)\cdot u(t-nT)\right\|_{W_2^n}^2}{\left\|\varphi(t)*u(t) - T\sum_n \varphi(nT)\cdot u(t-nT)\right\|_{L_2}}, \quad (7)$$

and $u(t)$ is given in Lemma 1.

The proof is given in [8].

Theorem 1 enables one to predict the maximum representation error induced by the sampling process. Furthermore, relying on the vector-like interpretation of Fig. 2, it can be shown that reducing the maximum potential approximation error to its minimum possible value is achieved by an orthogonal projection, thus optimal. This result suggests that the optimal discrete approximation scheme of the inner product does not necessarily require the sampled version of the basis function $\varphi(t)$ itself. Instead, it involves yet another sequence, arising from the orthogonal projection, as depicted in the next theorem..

Theorem 2: Let $\varphi(t) \in L_2$ be a known function. Given a sampling interval T, one can find an optimal sequence $b\,[n] \in l_2$ that minimizes B with respect to the inequality,

$$\forall f \in W_2, \quad \left| \langle f, g \rangle_{L_2} - \langle S_T f, b \rangle_{l_2} \right| \le B \cdot \|f\|_{L_2}, \qquad (8)$$

Here S_T is the sampling operator with interval T, $b[n]$ is derived by finding the orthogonal projecting, in the Sobolev sense, of $\varphi^* = \varphi * u$ onto $Span\{u(t-nT)\}_n$ (Fig. 3) where $u(t)$ is given in Lemma 1 and B is given in Lemma 1 by replacing $\varphi(nT)$ with $b[n]$.

The proof is given in [8].

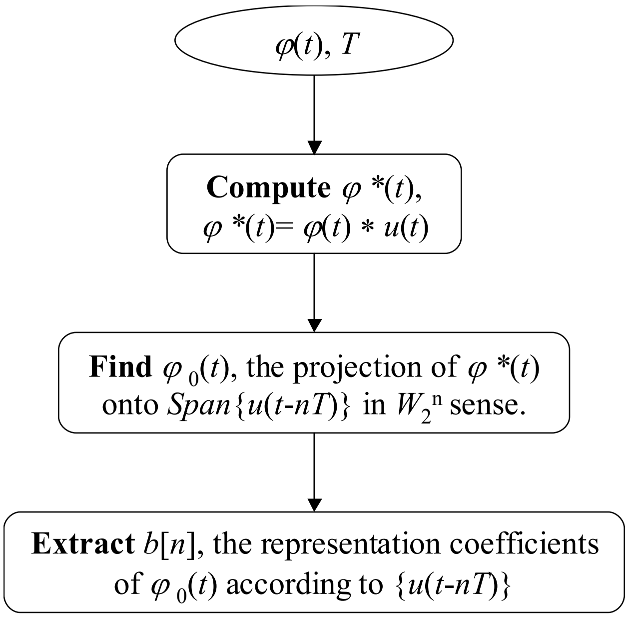


Figure 3: Optimal approximation of $\langle f, \varphi \rangle$ within the digital domain. Extracting $b[n]$ as shown here, and applying $\langle S_T\,\mathrm{f}, b \rangle$ would yield the minimum possible upper bound for the approximation error.

Both Theorem 1 and Theorem 2 can be extended to images as described in [8].

5. Examples

Example 1: Sampling a Gaussian. Suppose one wishes to find the inner product of a continuous-time signal with a normalized Gaussian. Having the sampled version of the signal, it is necessary to apply the discrete approximation scheme described herein. By setting,

$$\varphi(t) = \pi^{-1/4} e^{-t^2/2}, \qquad (9)$$

and applying Theorem 1, one can predict the maximum approximation error induced by the sampling process, shown in Fig. 4.

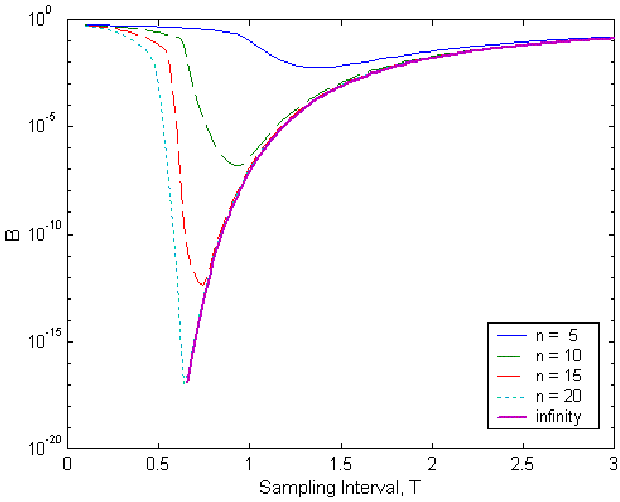


Figure 4: Upper bounds on the approximation error of $\langle f, \varphi \rangle$ by their corresponding sampled versions. Here φ is a normalized Gaussian. The upper bound is given by $B \cdot \|f\|_{L2}$. Shown are upper bounds where the admissible functions, f, are Sobolev functions of several orders (n = 5,10,15,20 and infinity).

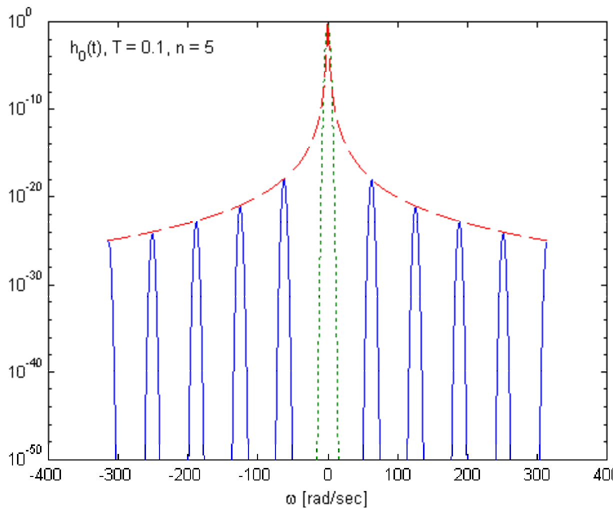


Figure 5: The worst-case scenario for approximating $\langle f, \varphi \rangle$ by their sampled versions. φ is a normalized Gaussian. The upper bound for the approximation error is $B \cdot \|f\|_{W2}$ where the function f achieving it is shown as well (solid). Here, a sampling interval of $T = 0.1$ is considered as well as admissible Sobolev functions, f, of order $n = 5$. Also shown is the corresponding $U(\omega)$.

For a given sampling interval, T, B corresponds to the worst-case scenario (i.e., the approximation error would be as large as possible). In such a case

$f(t) = \varphi^*(t) - S_T^*\{T\cdot\varphi(nT)\}$. As evident from Fig. 4, smaller sampling intervals than a certain threshold give rise to an asymptotic value of $B = 1$. It so happens that $\varphi(t)$ is effectively bandlimited for those sampling intervals, and the corresponding worst-case-scenario function would be then orthogonal to $\varphi(t)$ within the analog domain, which is not a practical situation.
The worst-case scenario for admissible Sobolev functions of order $n = 5$ and for a sampling interval of $T = 0.1$ is shown in Fig. 5.

Example 2: Suppose one wishes to determine whether two images are different or not. i.e., it is required to approximate the representation coefficients according to a set of basis images. Assuming that one of these basis images is the Gaussian, its representation coefficient can be calculated for various sampling intervals, as shown in Fig. 6. The key point, however, is that utilizing the abovementioned results; one can also extract the maximum potential approximation error induced by the sampling process in advance. Based on that information, a proper decision can then be made. It is evident from Fig. 5, that a sampling interval of $T = 1$ is insufficient for approximating the representation coefficient of the original images with regard to the Gaussian image. $T = 0.5$ is however sufficient, and there is no need to consider smaller sampling intervals such as $T = 0.1$.

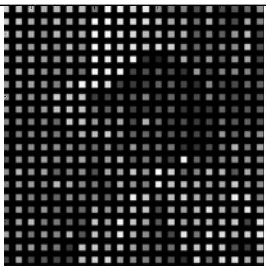
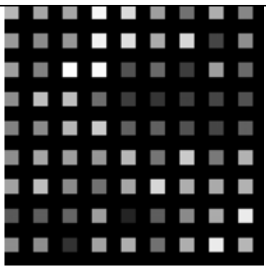
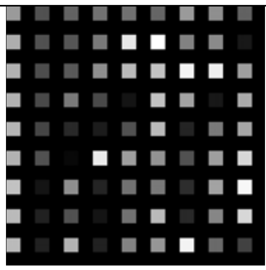
Sampling Interval	**Images**		**Discrete Approximation of the Inner Product with a Gaussian**	**Maximum Potential Error**
Original				
***T* = 0.1**			15.18 vs. 16.18	± 0.268
***T* = 0.5**			14.64 vs. 15.73	± 0.61
***T* = 1**			14.89 vs. 15.68	± 0.857

Figure 6: An example, utilizing Theorem 1. It is evident, that a sampling rate of $T = 1$ is insufficient for approximating the representation coefficient of the original images with regard to the Gaussian image. $T = 0.5$ is however sufficient, and there is no need to consider smaller sampling intervals such as $T = 0.1$.

5. Conclusions

The effect of sampling on non-bandlimited signals and images has been studied. In particular, the extent to which algebraic properties are preserved under sampling was investigated. The approach taken in this work arises from a signal representation point of view, in which an unknown continuous-time signal is to be represented by a set of continuous-time, analytically known, basis functions. In such a situation, both the sampled signal and the basis functions give rise to a discrete approximation scheme; the original inner product within the analog domain is approximated by an inner product of two sequences within the discrete domain.

By describing the sampling process as a bounded linear operator of two Hilbert spaces, a vector-like interpretation for this discrete approximation scheme has been derived. This interpretation enables one to determine a tight upper bound on the ensued approximation error, $B \cdot \|f\|$. This bound, B, is dependant on the basis function at hand, the sampling interval and the smoothness of the unknown function f. The only constraint imposed is that the functions to be sampled would be smooth. No constraints of bandlimited functions were assumed. Some examples were considered for both signals and images and several applications were suggested.

Furthermore, a new discrete approximation scheme for the inner product has been proposed, relying on the vector-like interpretation abovementioned. It has been shown that reducing the maximum potential approximation error to its minimum possible value is achieved by an orthogonal projection, thus optimal, Following this result, it has been further shown that the optimal discrete approximation scheme does not necessarily require the sampled version of the basis function itself. Instead, it involves yet another sequence, arising from the orthogonal projection previously mentioned.

The theorems presented in this work enable one to determine the maximum potential representation error induced by the sampling process regardless of the signals or images to be sampled. The results are applicable to signal processing applications, where digital signal representation of analog signals is required, based on their sampled versions (e.g., medical applications, data storage and data mining).

6. Acknowledgements

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