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A Fuzzy Description of the Henon Chaotic Map

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Abstract: - In this paper it is described how chaotic dynamics could be generated by means of fuzzy systems. The approach is based on a linguistic description of chaotic phenomena, that can be easily related to a fuzzy system design. It allows building chaotic generators,(useful in several industrial application in which randomness is needed) by means of few fuzzy sets and using a small number of fuzzy rules. Fuzzy descriptions of the well-known two dimensional discrete chaotic map is therefore introduced, denoting a good degree of approximation together with an easy design of the system obtained in a very intuitive way.

Key-Words: - Fuzzy Systems, Fuzzy Modelling, Chaotic Systems, Discrete Maps

1. Introduction

The possible interactions between fuzzy logic and chaos theory has been explored since the eighties, but these explorations has been carried on mainly in three directions: the fuzzy control of chaotic systems [1], [2], the definition of an adaptive fuzzy system by data from a chaotic time series [3], and the study of the theoretical relations between fuzzy logic and chaos [4]. We shall follow no one of these approaches, but, taking as starting point the work in [5], that of generating fuzzy systems which exhibits a chaotic behaviour via a linguistic descriptions of chaotic dynamics.

By giving a linguistic description of a chaotic system and by translating this description in a fuzzy model we achieve two results. The first one is to obtain fuzzy chaotic systems with desired characteristics and denoting an improved robustness to parameters changes. The second is to show that a simple and clear fuzzy system with few fuzzy sets and few rules is able to be a good model of a complex and cryptic chaotic system.

Another remarkable aspect to take in consideration is that there are several microcontrollers having a fuzzy core and therefore a fuzzy description of chaos can allow a deeper use of complexity techniques in several industrial applications without using expensive dedicated approaches.

2. Fuzzy and Chaos

During the last decade, the study of chaos has become increasingly important among physicist and engineers [6], due to the large number of its possible application. But what is “chaos”? Firstly were considered “chaotic” all those behaviours in some sense unpredictable due to the inadequate feature of measurement methodology (e.g. weather evolution). Nevertheless, technological improvements demonstrated that long term prediction of certain phenomena fails for their intrinsic complexity (highly non linear behaviour) and not for computational limitation. This quasi-random behaviour has been observed even in simple nonlinear systems, which demonstrated to be very sensitive

to changes of parameters. What was initially considered only as a “curious phenomenon”, nowadays is found everywhere in nature, showing a chaotic feature of our physical world. But the random behaviour of a deterministic system may also have some useful and surprising application in cryptography [7], signal processing [8] and, more generally, in most fields of industrial process control.

The peculiarities of a chaotic system can be listed as follows:

1. Strong dependence of the behavior from initial conditions
2. Sensitiveness to the changes of system parameters

3. Heavy presence of harmonics in the signals
4. Fractal dimension of space state trajectories
5. Presence of a stretch direction, represented by a positive Lyapunov exponent[9].

The latter can be considered as an "index" who quantifies a chaotic behavior.

Famous artificial chaotic systems are Chua's circuit [10], the Duffing oscillator [11] and the Roessler system, which can be represented as third order nonlinear autonomous systems. However chaotic dynamics can also be generated by simple discrete maps, like the logistic map:

$$x(k+1) = ax(k)(1 - x(k)) \quad (1)$$

or the Henon map:

$$\begin{cases} x(k+1) = y(k) + 1 - ax^2(k) \\ y(k+1) = bx(k) \end{cases} \quad (2)$$

In [14] it has been described how the behaviour of the mono-dimensional map (1) can be modeled by means of four linguistic variables [12]: *x(k), x(k+1)*, and the associated uncertainty quantities *d(k)* and *d(k+1)*, which play an important role in stretching and folding typical of chaotic phenomena. Membership functions of these variables are depicted in Fig. 1.

In next section this approach will be extended to the two-dimensional map (2), avoiding some of the drawbacks related to the increasing of the order (e.g. high number of fuzzy sets and rules).

3. Fuzzy Modeling of the Henon Map

To model the evolution of a chaotic signal *x(.)* two variables need to be considered as inputs: the 'center' value *x(k)* , which is the nominal value of the state x(k) at the step k and the uncertainty d(k) on the center value. In terms of fuzzy description, this means that the model contains, as previously said, four linguistic variables. In the case of the logistic map model [14], it has been adopted a two inputs and two outputs fuzzy model ([*x*(*k*+1),*d*(*k*+1)]=*F*(*x*(*k*),*d*(*k*))) with the Mamdani implication, the center-of-sums defuzzification method and the product as t-norm [13]. The Henon map, still introduced in section 2, can be considered in some sense the two dimensional extension of the logistic map.Here we rewrite again the state equations:

$$\begin{cases} x(k+1) = y(k) + 1 - ax^2(k) \\ y(k+1) = bx(k) \end{cases} \quad (2)$$

For some values of parameters *a* and *b,* the discrete state space plot (or *Poincarè* map) denotes a fractal-like limit set typical of a chaotic system (see Fig. 2). The time evolution of the state variable *x* is depicted in fig. 3. The same behaviour can be observed for *y*, which is proportional to a one-sample delayed sequence of *x*, as it can be seen from equation (2).

An approach similar to that described in [14] could be attempted. In this case a function $F(x(k), d_x(k), y(k), d_y(k)) = (x(k+1), d_x(k+1), y(k+1), d_y(k+1))$ should be modelled, taking into account uncertainties d_x and d_y for each one of the two variables. However this fact may lead to a quite complex definition of the qualitative fuzzy model, if compared with the analytical description of the system. In order to avoid these complications, we assume to have only the uncertainty d_x, considering that it can influence also y through the second equation of (2) in a linear way. Being *y(k+1)* proportional to *x(k)*, membership function of *x* and *y* could be chosen as identical. However the light influence of *y(k)* on *x*(*k*+1) with given parameters (*a*=1.4 and *b*=0.3) suggest us to use fewer fuzzy sets for *y(k)*, which acts only if its absolute value is high. The choice for *x(k)* is therefore similar to that adopted in the logistic map, because of a similar parabolic behaviour (but different range). Even in this case there are two equilibrium points (E_1≡(-1.31,-0.34), E_2≡(0.63, 0.19)), both of them unstable.

The fuzzy sets associated to the linguistic values are shown in Fig. 3; they have been constructed in such a way that E_2≡(0.63, 0.19) is between the fuzzy set M and the fuzzy set L of *x* and *y*, whose number is in this case different. In fact, for *y* only three fuzzy sets are adopted, being remarkable for the evolution of *x* (first equation of (2)) only small or large values of *y* (medium values, close to zero, are in this case neglected). Fuzzy sets of the uncertainty d_x are exactly the same as considered in [14], but with different ranges.

Qualitatively *x* evolves similarly to the logistic map: *x* tends to move towards its value in E_2 , until it begins to oscillate around that point. When oscillations reach the neighbourhood of E_1, this perturbs the trajectory moving it again towards E_2. To this aim, when the influence of *y*(*k*) on *x*(*k*+1) is light (approximately when *y*(*k*) is M), it is possible to keep the same rules still used for the logistic map in [14] (see table 1).

The influence becomes relevant when y(*k*) is L or S, even if limited to the transition L-VL and S-Z of *x*. Therefore, when y(*k*) is S, the new rules that have to be added are the following:

HS_1 if $x(k)$ is Z and $d(k)$ is L and $y(k)$ is S then $x(k+1)$ is Z (instead of S)
HS_2 if $x(k)$ is M and $d(k)$ is VL and $y(k)$ is S then $x(k+1)$ is L (instead of VL)

The complete set of rules for this case is reported in table 2. The differences with respect to table 1 are underlined. The action of *y* decreases $x(k+1)$ from S to Z or from VL to L.

On the other side, when $y(k)$ is L, only increasing transitions (from L to VL or from S to Z) take place. Therefore, the new rules to be added are the following:

HS_3 if $x(k)$ is Z and $d(k)$ is Z and $y(k)$ is L then $x(k+1)$ is S (instead of Z)
HS_4 if $x(k)$ is Z and $d(k)$ is S and $y(k)$ is L then $x(k+1)$ is S (instead of Z)
HS_5 if $x(k)$ is Z and $d(k)$ is M and $y(k)$ is L then $x(k+1)$ is S (instead of Z)
HS_6 if $x(k)$ is Z and $d(k)$ is VL and $y(k)$ is L then $x(k+1)$ is VL (instead of L)
HS_7 if $x(k)$ is M and $y(k)$ is L then $x(k+1)$ is VL (instead of L)
HS_8 if $x(k)$ is L and $d(k)$ is VL and $y(k)$ is L then $x(k+1)$ is S (instead of Z).

$x(k)/d(k)$	Z	S	M	L	VL
Z	Z/Z	Z/M	Z/M	S/VL	L/L
S	M/Z	M/M	M/M	M/VL	L/S
M	L/Z	L/M	L/M	L/VL	VL/S
L	M/Z	M/M	M/M	M/VL	Z/S
VL	Z/Z	Z/M	Z/M	Z/VL	Z/L

Table 1. The set of rules for the evaluation of x(k+1) and d(k+1) when y(k) is M.

$x(k)/d(k)$	Z	S	M	L	VL
Z	Z/Z	Z/M	Z/M	**Z**/VL	L/L
S	M/Z	M/M	M/M	M/VL	L/S
M	L/Z	L/M	L/M	L/VL	**L**/S
L	M/Z	M/M	M/M	M/VL	Z/S
VL	Z/Z	Z/M	Z/M	Z/VL	Z/L

Table 2. The set of rules for the evaluation of x(k+1) and d(k+1) when y(k) is S. The differences are underlined

The complete set of rules for this case is reported in table 3. The changes made with respect to table 1 are underlined even in this case.

In order to complete the whole set of rules for this fuzzy system, the dynamic of $y(k)$ has to be considered. Due to the second equation of (2), these simple rules can be added:

HS_9 if $x(k)$ is Z then $y(k+1)$ is S
HS_{10} if $x(k)$ is S then $y(k+1)$ is S
HS_{11} if $x(k)$ is M then $y(k+1)$ is M
HS_{12} if $x(k)$ is L then $y(k+1)$ is L
HS_{13} if $x(k)$ is VL then $y(k+1)$ is L

These statements can be summarized in table 4.

$x(k)/d(k)$	Z	S	M	L	VL
Z	**S**/Z	**S**/M	**S**/M	S/VL	**VL**/L
S	M/Z	M/M	M/M	M/VL	L/S
M	**VL**/Z	**VL**/M	**VL**/M	**VL**/VL	VL/S
L	M/Z	M/M	M/M	M/VL	**S**/S
VL	Z/Z	Z/M	Z/M	Z/VL	Z/L

Table 3. The set of rules for the evaluation of x(k+1) and d(k+1) when y(k) is L. The differences are underlined

$x(k)$ / $y(k)$	S	M	L
Z	S	S	S
S	S	S	S
M	M	M	M
L	L	L	L
VL	L	L	L

Table 4. The set of rules which allows to evaluate y(k+1). It depends only on x(k).

The so-designed fuzzy system is now completed and its dynamic evolution can be derived. By choosing as initial condition $[x(0)\ \ y(0)\ d(0)]$ = [0.01 0.01 0.01], it is possible to obtain the behaviour of $x(k)$ (figure 5) and the space-state plot representing both variables *x* and *y* (figure 6).

4. Conclusions

In this paper a qualitative approach for fuzzy modelling of chaotic dynamics has been discussed. This analysis has pointed out several facts regarding both fuzzy logic and chaos theory:

1. Simple fuzzy systems are able to generate complex dynamics.
2. The precision in the approximation of the time series depends only on the number and the shape of the fuzzy sets for the *x*.

3. The analysis of a chaotic system via a linguistic description allows a better understanding of the system itself.

4. Accurate generator of chaos with desired characteristics can be built using fuzzy model.

5. Multi-dimensional chaotic maps in some cases do not need a large number of rules in order to be represented

Future researches on fuzzy modeling of chaotic systems may be developed in several directions. Qualitative analysis should evolve into the automatic design of the fuzzy system improving the precision of the resulting model. These results could be used for a great number of applications, above all the generation of chaotic signals for cryptographic purposes, which could require well-defined statistic properties of the signal. Moreover, several microcontrollers have a fuzzy core and therefore a fuzzy description of chaos can allow a deeper use of complexity techniques in several industrial applications without using expensive dedicated approaches (e.g. industrial control and optimization applications).

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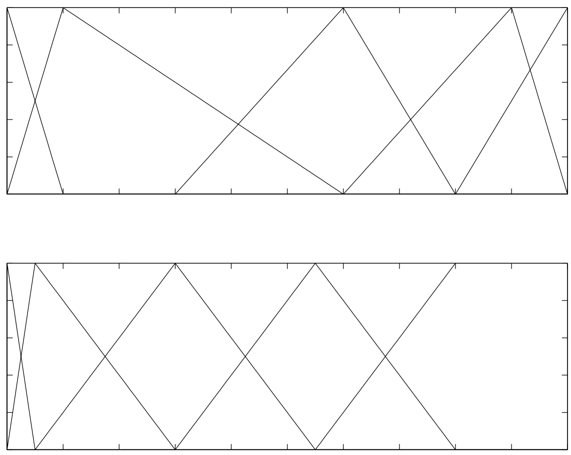

Fig. 1. The fuzzy sets for x (upper) and d (lower) for the model of logistic map.

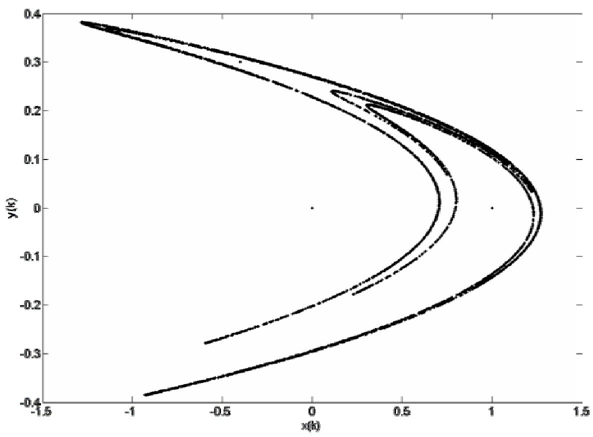

Fig. 2. Henon Map state space plot (parameters a=1.4, b= 0.3)

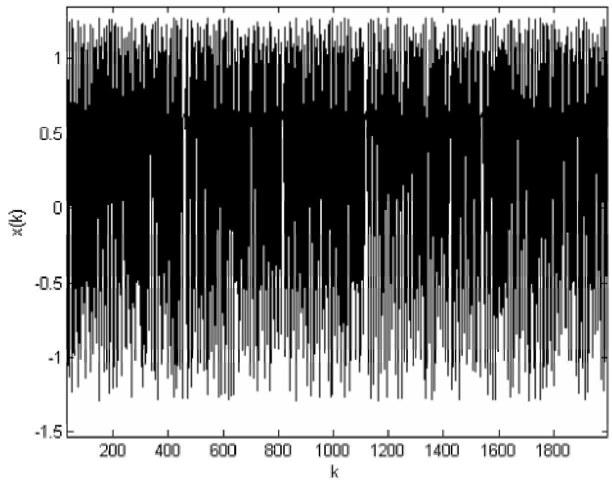

Fig. 3. The chaotic time series generated by the Henon map (parameters a=1.4, b= 0.3)

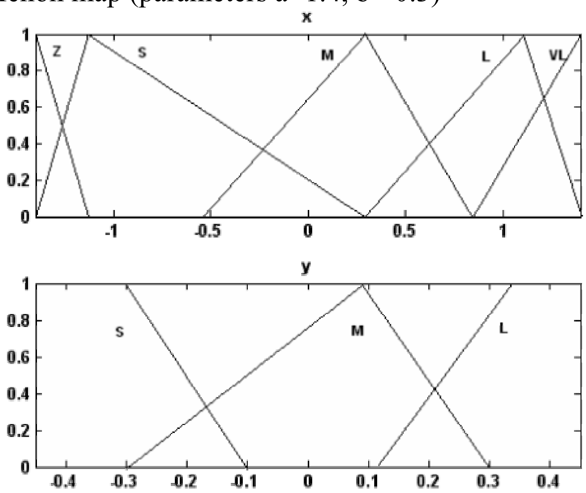

Fig. 4. The fuzzy sets for *x* (upper) and *y* (lower) in the qualitative Henon map.

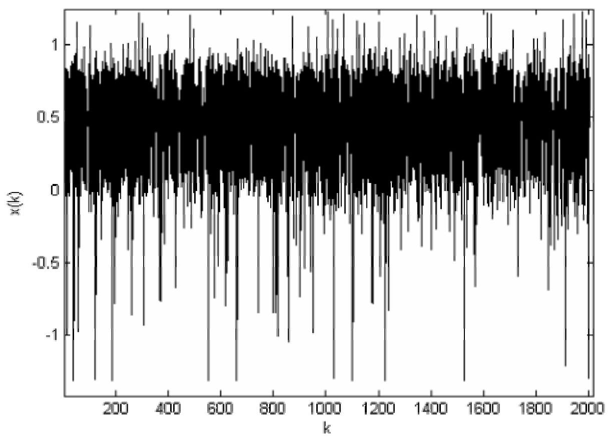

Fig. 5. Time evolution of state variable *x(k)* generated by the fuzzy system

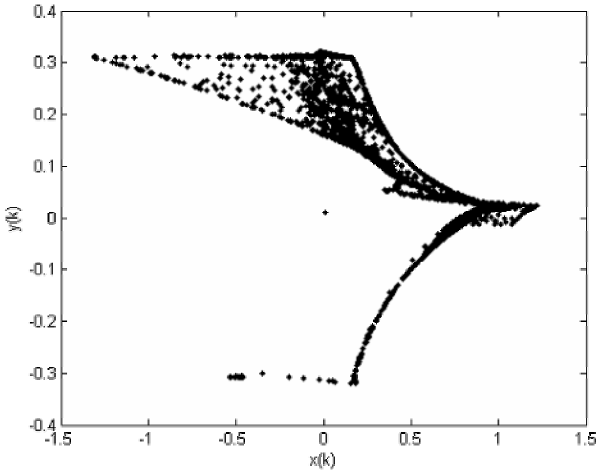

Fig. 6. Space-state plot of variables x(k) and y(k) generated by the fuzzy system