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Λ-operator ; a tool for solving ODEs

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Abstract: - The Λ-operator is applied to solve various kinds of ordinary differential equations. Using the Kuperschmidt-Wilson factorization of the Λ-operator, the operator representation of the Darboux transformation is given. Moreover, the degenerate condition for the non-spectral Darboux transformation is clarified.

Keywords: - Λ-operator, Kuperschmidt-Wilson factorization, Darboux transformation, KdV polynomials

1 Introduction

The formal pseud differential operator

$$\Lambda(u) = \left(\frac{d}{dx}\right)^{-1}\left(\frac{1}{2}u' + u\frac{d}{dx} - \frac{1}{4}\frac{d^3}{dx^3}\right)$$

is called the Λ-operator, where $u = u(x)$ is a differentiable function of the variable x, and u' is its derivative. Note that our discussion is valid for both the real variable and the complex variable. We refer the reader to [6] and [7] for the precise meaning of $\Lambda(u)$.

The Λ-operator is usually used in connection with the symmetric approach to the KdV equation. See [3] for this fact. In this paper, we use the Λ-operator to solve a kind of ordinary differential equations (ODEs). Problems related to the spectral theory of the 2nd order ordinary differential operator were mainly considered in the previous works of the present author [4], [6], [7]. In this paper, apart from those approach, we will discuss the feature of the Λ-operator as a tool of an algorithm for solving ODEs.

The contents of the present paper are as follows. §2 is devoted to the preliminaries. In §3, the Kuperschmidt-Wilson factorization of $\Lambda(u)$ is introduced, and the operator representation of the Darboux transformation is given. In §4, the non-spectral degenerate condition with respect to the Darboux transformation is clarified.

2 Preliminaries

Let $Z_0(u) = 1$, and define $Z_n(u)$ for the natural number n by the recursion relation

$$(1) \qquad Z_n(u) = \Lambda(u)Z_{n-1}(u),\ n = 1, 2, \cdots .$$

Then, it is known that $Z_n(u)$ are the differential polynomials in u and its derivatives [4, Lemma 3.1, p621]. We call them the KdV polynomials. Let $V(u)$ be the linear span of all KdV polynomials over the complex number field $\mathbb{C}$. If $\dim V(u) = n + 1 < \infty$, the potential $u(x)$ is called the algebro-geometric potential and the Λ-rank of $u(x)$ is defined by

$$\mathrm{rank}_\Lambda u(x) = \dim V(u) - 1 = n.$$

If $\mathrm{rank}_\Lambda u(x) = n$, then, it is shown in [6, Lemma 5, p416] that $Z_0(u), Z_1(u), \cdots, Z_n(u)$ are the basis of the vector space $V(u)$. Therefore there uniquely exists the constants c_ν, $0 \le \nu \le n$ such that

$$(2) \qquad Z_{n+1}(u) = \sum_{\nu=0}^{n} c_\nu Z_\nu(u).$$

Based on the identity (2), the M-function is defined by

$$(3) \quad M = M(x;u) = Z_n(u) - \sum_{\nu=1}^{n} c_\nu Z_{\nu-1}(u).$$

By (3) and (1), we have

$$\begin{aligned}\Lambda(u)M(x;u) \\ &= Z_{n+1}(u) - \sum_{\nu=1}^{n} c_\nu \Lambda(u) Z_{\nu-1}(u) \\ &= Z_{n+1}(u) - \sum_{\nu=1}^{n} c_\nu Z_\nu(u) = -c_0\end{aligned} \tag{4}$$

By (4), it follows that $F = M(x;u)$ solves the 3rd order ODE

$$\frac{1}{2}u'(x)F + u(x)\frac{dF}{dx} - \frac{1}{4}\frac{d^3F}{dx^3} = 0. \tag{5}$$

By the way, if $f(x)$ and $g(x)$ are the solutions of the 2nd order ODE

$$\frac{d^2y}{dx^2} = u(x)y, \tag{6}$$

then the product $F = f(x)g(x)$ turns out to solve the 3rd order ODE (5). For this fact, see [10, p298], in which this is refered as Appell's lemma. By direct calculation, one verifies the identity

$$W[f^2, fg, g^2] = 2W[f,g]^2,$$

where $W[f,\cdots,g]$ is the Wronskian. Hence, if $f(x)$ and $g(x)$ are linearly independent, then, it follows immediately that $f(x)^2$, $f(x)g(x)$ and $g(x)^2$ are also linearly independent. Hence the M-function $M(x;u)$ is expressed as the linear combination

$$M(x;u) = \alpha_1 f(x)^2 + \alpha_2 f(x)g(x) + \alpha_3 g(x)^2.$$

Conversely, we want to construct the solution of the 2nd order ODE (6) from the solution of the 3rd order ODE (5). However this is not possible in general. That is, suppose that $F(x)$ is a solution of (5). Then, though we do not know definitely the fundamental system of solutions $f(x)$ and $g(x)$ of (6) themselves, but we can express $F(x)$ as the linear combination

$$F(x) = \alpha_1 f(x)^2 + \alpha_2 f(x)g(x) + \alpha_3 g(x)^2.$$

If the perfect square condition

$$\alpha_2^2 - 4\alpha_1\alpha_3 = 0 \tag{7}$$

is fulfilled, then

$$\alpha_1\xi^2 + \alpha_2\xi\eta + \alpha_3\eta^2 = (\sqrt{\alpha_1}\xi + \sqrt{\alpha_3}\eta)^2 \tag{8}$$

follows, i.e., the left hand side of (8) is the perfect square quadratic form. That is, we have

$$F(x) = (\sqrt{\alpha_1}f(x) + \sqrt{\alpha_3}g(x))^2.$$

Then

$$h(x) = \sqrt{F(x)} = \sqrt{\alpha_1}f(x) + \sqrt{\alpha_2}g(x)$$

is a solution of the 2nd order ODE (6).

One can choose $f(x)$ and $g(x)$ as the solutions which satisfy the initial condition

$$\begin{pmatrix} f(a) & g(a) \\ f'(a) & g'(a) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for some $x = a$. Then, by direct calculation, we have

$$F(a) = \alpha_1,\ F'(a) = \alpha_2,\ F''(a) = 2\alpha_1 u(a) + 2\alpha_3.$$

Hence the condition (7) is equivalent to the condition

$$\begin{aligned}\Delta(a) &= F'(a)^2 \\ &\quad - 2F(a)F''(a) + 4u(a)F(a) = 0.\end{aligned}$$

$\Delta(a)$ seems to depend on a. But, by direct calculation, one easily verifies

$$\begin{aligned}\frac{d}{da}\Delta(a) \\ &= -2FF''' + 4u'F^2 + 8uFF' \\ &= -2F(2u'F + 4uF') + 4u'F^2 + 8uFF' \\ &= 0.\end{aligned}$$

Thus we proved the following lemma which corresponds to the converse of Appell's lemma.

Lemma 1. *Let $F(x)$ be a solution of the 3rd order ODE (5) such that $\Delta(a) = 0$ for some $x = a$, then $\Delta(x) \equiv 0$ holds and $\sqrt{F(x)}$ is a solution of the second order ODE (6).*

Applying this to the M-function $M(x;u)$, we have the following.

Proposition 1. *Assume that $\mathrm{rank}_\Lambda u = n < \infty$ and $\Delta_0 = 0$ holds, where*

$$\begin{aligned}\Delta_0 &= M'(x;u)^2 \\ &\quad - 2M(x;u)M''(x;u) + 4u(x)M(x;u)^2.\end{aligned} \tag{9}$$

Then $h(x) = \sqrt{M(x;u)}$ is a solution of the 2nd order ODE (6).

Proceedings of the 5th WSEAS/IASME Int. Conf. on SYSTEMS THEORY and SCIENTIFIC COMPUTATION, Malta, September 15-17, 2005 (pp163-168)

This implies that the assumption of Proposition 1 is fullfilled then the 2nd order ODE (6) has a solution expressed explicitly in terms of the coefficient $u(x)$.

Definition 1. The solution $F(x)$ of the 3rd order ODE (5) is called the perfect square solution, if $\Delta(a) = 0$. Moreover, the M-function $M(x;u)$ is called the perfect square M-solution, if the condition $\Delta_0 = 0$ holds, where Δ_0 is defined by (9).

3 Kuperschmidt-Wilson factorization

Let $f(x)$ and $g(x)$ be the fundamental system of solutions of the 2nd order ODE (5). Define the function $w(x;\alpha)$, $\alpha \in \mathbb{C} \cup \{\infty\}$ by

$$w(x;\alpha) = \begin{cases} f(x) + \alpha g(x), & \alpha \in \mathbb{C} \\ g(x), & \alpha = \infty. \end{cases}$$

Let

$$v(x;\alpha) = \frac{d}{dx} \log w(x;\alpha).$$

The Darboux transformation $u^*(x;\alpha)$ of the potential $u(x)$ is defined by

$$u^*(x;\alpha) = u(x) - 2\frac{d}{dx} v(x;\alpha). \tag{10}$$

The transformation (10) was first introduced by J. G. Darboux [2], and used by M. M. Crum [1] as an algorithm for adding or removing eigenvalues of the Sturm-Liuville operator. Recently, the present author studied it by an algebraic method [4], [5], [6], [7], [8].

On the other hand, in [3], the factorizations

$$\Lambda(u) = \frac{1}{4}\left(\frac{d}{dx}\right)^{-1} \cdot B_+(\alpha) \cdot \left(\frac{d}{dx}\right) \cdot B_-(\alpha) \tag{11}$$

and

$$\Lambda(u^*(x;\alpha)) = \frac{1}{4}\left(\frac{d}{dx}\right)^{-1} \cdot B_-(\alpha) \cdot \left(\frac{d}{dx}\right) \cdot B_+(\alpha) \tag{12}$$

are shown, where

$$B_\pm(\alpha) = \pm\frac{d}{dx} + 2v(x;\alpha).$$

In [5], applying the factorizations (11) and (12), it is shown that the identity

$$B_+(\alpha) Z_n(u^*(x;\alpha)) = B_-(\alpha) Z_n(u(x))$$

holds for $n = 0, 1, 2, \cdots$. This implies that the Darboux transformation (10) can be expressed formally as

$$u^*(x;\alpha) = B_+^{-1}(\alpha) B_-(\alpha) u(x).$$

Of course, the operator B_+^{-1} is not univalent correspondence, but determined modulo $W(\alpha)$, where $W(\alpha) = \mathbb{C}w(x;\alpha)$ is the 1-dimensional vector space. However, for a formal treatment of the Darboux transformation, sometimes a formal expression like this is quite covenient. More generally, we have the following.

Proposition 2. *The operator representation*

$$Z_m(u^*(x;\alpha)) = \frac{1}{2} B_+^{-1}(\alpha) B_-(\alpha) \Lambda(u)^{m-1} u(x).$$

holds for all $m \geq 1$.

In what follows, we assume that $\mathrm{rank}_\Lambda u(x) = n < \infty$ and the relation (2) always holds. By (2), we have

$$B_-(\alpha) Z_{n+1}(u) = \sum_{\nu=0}^{n} c_\nu B_-(\alpha) Z_\nu(u). \tag{13}$$

Operate the both sides of the equality (13) with the operator $\frac{1}{4}\left(\frac{d}{dx}\right)^{-1} \cdot B_-(\alpha) \cdot \left(\frac{d}{dx}\right)$, then, by the factorization (12), we have

$$\Lambda(u^*(x;\alpha)) Z_{n+1}(u^*(x;\alpha) = \sum_{\nu=0}^{n} c_\nu \Lambda(u^*(x;\alpha)) Z_\nu(u^*(x;\alpha)).$$

By (1), we have

$$Z_{n+2}(u^*(x;\alpha)) = \sum_{\nu=0}^{n} c_\nu Z_{\nu+1}(u^*(x;\alpha)). \tag{14}$$

Using this identity, in [5, pp.12-13], the inequality concerned with the Λ-rank

$$\mathrm{rank}_\Lambda u(x) - 1 \leq \mathrm{rank}_\Lambda u^*(x;\alpha) \leq \mathrm{rank}_\Lambda u(x) + 1.$$

is obtained.

On the other hand, the relation (2) is nothing but the n-th order stationary KdV equation. At the same time, the relation (14) is the $(n+1)$-th stationary KdV equation. Hence, the

Proceedings of the 5th WSEAS/IASME Int. Conf. on SYSTEMS THEORY and SCIENTIFIC COMPUTATION, Malta, September 15-17, 2005 (pp163-168)

Darboux transformation $u(x) \longrightarrow u^*(x;\alpha)$ can be regarded as the transformation between the n-th and $(n+1)$-th stationary KdV equations. Thus, the relation (14) can be regarded as the transformation formula of the higher order KdV equation with respect to the Darboux transformation. The relation (2) characterizes the potential $u(x)$. However, if

$$\mathrm{rank}_\Lambda u^*(x;\alpha) = \mathrm{rank}_\Lambda u(x) - 1 \tag{15}$$

holds, the transformed potential $u^*(x;\alpha)$ satisfies much more simpler relation

$$Z_n(u^*(x;\alpha)) = \sum_{\nu=0}^{n-1} c'_\nu Z_\nu(u^*(x;\alpha)).$$

So, in the following part of this paper, we will discuss the condition for (15).

4 Non-spectral degenerate condition

In this section, we consider how solutions of the 2nd order ODE (6) and the 3rd order ODE (5) are transformed when the coefficient $u(x)$ is transformed by the Darboux transformation. In what follows, we consider the Darboux transformation by the solution

$$h(x) = \sqrt{M(x;u)}, \tag{16}$$

where $M(x;u)$ is the perfect square M-solution. In this case, there exists $\alpha_0 \in \mathbb{C} \cup \{\infty\}$ such that

$$\frac{d}{dx}\log h(x) = v(x;\alpha_0),$$

and we have

$$\begin{aligned} &u^*(x;\alpha_0) \\ &= u(x) - 2\frac{d}{dx}v(x;\alpha_0) \\ &= u(x) - \frac{d^2}{dx^2}\log M(x;u). \end{aligned} \tag{17}$$

Now we want to clarify the condition for

(18)
$$\mathrm{rank}_\Lambda u^*(x;\alpha_0) = \mathrm{rank}_\Lambda u(x) - 1 = n-1.$$

In [7], the spectral degenerate condition is obtained. In what follows, we briefly explain it. Let us consider the 2nd order ODE

$$\frac{d^2y}{dx^2} = (u(x) - \lambda)y. \tag{19}$$

We assume that $\mathrm{rank}_\Lambda u(x) = n$ and the relation (2) holds. Define the coefficients $\alpha_\nu^{(m)}$ by

(20)
$$\alpha_\nu^{(m)} = \begin{cases} 1, & \nu = m \\ \alpha_{\nu-1}^{(m-1)} + \alpha_\nu^{(m-1)}, & 1 \le \nu \le m-1 \\ \frac{(2m)!}{2^{2m}(m!)^2}, & \nu = 0. \end{cases}$$

Then, the following expansion formula

$$Z_m(u(x) - \lambda) = \sum_{\nu=0}^{m} (-1)^{m-\nu} \alpha_\nu^{(m)} Z_\nu(u(x))\lambda^{m-\nu}$$

holds. Moreover define the polynomials $a_\nu(\lambda)$, $\nu = 1, \cdots, n$ in λ by

(21)
$$a_\nu(\lambda) = -\alpha_\nu^{(n+1)}\lambda^{n-\nu+1} + \sum_{\mu=\nu}^{n} \alpha_\nu^{(\mu)} c_\mu \lambda^{\mu-\nu},$$

where c_μ are the constants appeared in the relation (2). Then $\mathrm{rank}_\Lambda(u(x)-\lambda) = n$ and the relation

$$Z_{n+1}(u(x)-\lambda) = \sum_{\nu=0}^{n} a_\nu(\lambda) Z_\nu(u(x)-\lambda)$$

follow. Therefore the M-function for $u(x) - \lambda$ (the spectral M-function) is defined by

(22)
$$\begin{aligned} M(x,\lambda;u) &= M(x;u-\lambda) \\ &= Z_n(u(x)-\lambda) - \sum_{\nu=1}^{n} a_\nu(\lambda) Z_{\nu-1}(u(x)-\lambda). \end{aligned}$$

Define the spectral discriminant $\Delta(\lambda;u)$ by

$$\begin{aligned} &\Delta(\lambda;u) = \\ &\quad M_x(x,\lambda;u)^2 - 2M(x,\lambda;u)M_{xx}(x,\lambda:u) \\ &\qquad + 4(u(x)-\lambda)M(x,\lambda;u)^2. \end{aligned}$$

If $\Delta(\lambda_0;u) = 0$, then the spectral M-function $F = M(x,\lambda_0;u)$ is the perfect square solution of the 3rd order ODE

$$\frac{1}{2}u'(x)F + (u(x)-\lambda_0)\frac{dF}{dx} - \frac{1}{4}\frac{d^3F}{dx^3} = 0.$$

Hence, $y = \sqrt{M(x,\lambda_0;u)}$ is the solution of the 2nd order ODE (19) for $\lambda = \lambda_0$. Define the Darboux transformation at $\lambda = \lambda_0$ by

$$u^*_{\lambda_0}(x) = u(x) - \frac{d^2}{dx^2}\log M(x,\lambda_0;u).$$

In [7, Theorem, p.959], the following degenerate condition is established.

Theorem 1. *$rank_\Lambda u^*_{\lambda_0}(x) = n-1$, if and only if λ_0 is the multiple root of the spectral discriminant $\Delta(\lambda;u)$.*

By Theorem 1, $u^*(x;\alpha)$ defined by (17) is degenerate, i.e., (18) holds, if and only if $\lambda = 0$ is the multiple root of the spectral discriminant $\Delta(\lambda;u)$. Though, in a sense, this is a kind of abstract nonsense, however, one can calculate it definitely in this case.

It is obvious that $\lambda = 0$ is the multiple root of the spectral discriminant $\Delta(\lambda)$, if and only if both the coefficients of the terms of the 0-th and the 1st degree vanish. The 0-th coefficient is nothing but Δ_0 defined by (9). Hence, vanishing of the 0-th coefficient is equivalent to perfect square condition. Therefore it is necessary to calculate the coefficient Δ_1 of the term of the 1st degree, i.e.,

$$\Delta(\lambda;u) = \Delta_0 + \Delta_1\lambda + O(\lambda^2),$$

where $O(\lambda^2)$ denotes the terms whose degree are higher than 2. In the following, we carry out somewhat complicated calculation to express Δ_1 in terms of $Z_\nu(u(x)), \nu = 0,1,2,\cdots,n$. First we have the following.

Lemma 2. *Suppose $n \geq 2$, and let*

$$N(x;u) = Z_{n-1}(u(x)) - \sum_{\nu=2}^{n} c_\nu Z_{\nu-2}(u(x)).$$

Then

$$M(x,\lambda;u) = M(x;u) + N(x;u)\lambda + O(\lambda^2)$$

holds.

Proof. By (20), we have $\alpha^{(m)}_m = 1$ and

$$\begin{aligned}\alpha^{(m)}_{m-1} &= \alpha^{(m-1)}_{m-2} + \alpha^{(m-1)}_{m-1} = \alpha^{(m-1)}_{m-2} + 1\\ &= \alpha^{(m-2)}_{m-3} + \alpha^{(m-2)}_{m-2} + 1 = \alpha^{(m-2)}_{m-3} + 2\\ &\quad\cdots\cdots\cdots\cdots\\ &= \alpha^{(1)}_0 + m - 1 = \frac{1}{2} + m - 1 = m - \frac{1}{2}.\end{aligned}$$

Hence

$$Z_m(u(x)-\lambda) = Z_m(u(x)) - (m-\frac{1}{2})Z_{m-1}(u(x))\lambda + O(\lambda^2)$$

follows. On the other hand, by (21), we have immediately

$$a_\nu(\lambda;u) = c_\nu + (\nu+\frac{1}{2})c_{\nu+1}\lambda + O(\lambda^2)$$

for $\nu = 1,2,\cdots,n-1$. Moreover, for $\nu = n$, one verifies directly

$$a_n(\lambda;u) = c_n - (n+\frac{1}{2})\lambda.$$

Therefore, by (22), we have

$$\begin{aligned}M(x,\lambda;u) &=\\ &Z_n(u) - (n-\frac{1}{2})Z_{n-1}(u)\lambda + O(\lambda^2)\\ &+ ((n+\frac{1}{2})\lambda - c_n)(Z_{n-1}(u)\\ &- (n-\frac{3}{2})Z_{n-2}(u)\lambda + O(\lambda^2))\\ &+ \sum_{\nu=2}^{n-2}(c_\nu + (\nu+\frac{1}{2})c_{\nu+1}\lambda + O(\lambda^2))(Z_{\nu-1}(u)\\ &- (\nu-\frac{3}{2})Z_{\nu-2}(u)\lambda + O(\lambda^2))\\ &- c_1 - \frac{3}{2}c_2\lambda + O(\lambda^2)\\ &= Z_n(u) - \sum_{\nu=1}^{n-1} c_\nu Z_{\nu-1}(u)\\ &+ (-(n-\frac{1}{2})Z_{n-1}(u) + (n+\frac{1}{2})Z_{n-1}(u)\\ &+ (n-\frac{3}{2})c_n Z_{n-2}(u)\\ &- (\sum_{\nu=2}^{n-1}((\nu+\frac{1}{2})c_{\nu+1}Z_{\nu-1}(u)\\ &- (\nu-\frac{3}{2})c_\nu Z_{\nu-2}(u)) - \frac{3}{2}c_2)\lambda + O(\lambda^2)\\ &= M(x;u) + N(x;u)\lambda + O(\lambda^2)\end{aligned}$$

This completes the proof.

By the above lemma 2, we can calculate the coefficient of the term of the 1st degree. Thus we have the following lemma.

Lemma 3. *Suppose $n \geq 2$, then*

(23)
$$\Delta_1 = 2(M'N' - MN'' - M''N + 4uMN - 2M^2)$$

holds.

Proof. We have

$$\begin{aligned}
\Delta(\lambda; u) = \\
& M_x(x,\lambda;u)^2 - 2M(x,\lambda;u)M_{xx}(x,\lambda;u) \\
& \quad + 4(u(x)-\lambda)M(x,\lambda;u)^2 \\
&= (M' + N'\lambda + O(\lambda^2))^2 \\
& \quad - 2(M + N\lambda + O(\lambda^2))(M'' + N''\lambda \\
& \quad + O(\lambda^2)) + 4(u-\lambda)(M + N\lambda + O(\lambda^2))^2 \\
&= M'^2 - 2MM'' + 4uM^2 \\
& \quad + 2(M'N' - MN'' - M''N + 4uMN \\
& \quad - 2M^2)\lambda + O(\lambda^2) \\
&= \Delta_0 + \Delta_1\lambda + O(\lambda^2).
\end{aligned}$$

This completes the proof.

Δ_1 defined by (23) seems to depend on x. But, by direct calculation, we have

$$\begin{aligned}
\frac{d}{dx}\Delta_1 &= 2N(4u'M + 4uM' - M''') \\
& \quad + 2M(4uN' - N''') - 8MM' \\
&= 8M(\frac{1}{2}u'N + uN' - \frac{1}{4}N''') - 8MM' \\
&= 8M(\Lambda(u)N)' - 8MM'.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\Lambda(u)N &= \Lambda(u)(Z_{n-1}(u) - \sum_{\nu=2}^{n} c_\nu Z_{\nu-2}(u)) \\
&= Z_n(u) - \sum_{\nu=2}^{n} c_\nu Z_{\nu-1}(u) \\
&= M + c_1
\end{aligned}$$

Hence $(\Lambda(u)N)' = M'$ follows, i.e., Δ_1 does not depend on x.

Thus, by lemma 2 and lemma 3, we have the following theorem which is the main result of the present work.

Theorem 2. *Suppose that $2 \leq rank_\Lambda u(x) = n < \infty$ and $\Delta_0 = 0$. Then*

$$rank_\Lambda u^*(x;\alpha_0) = n - 1,$$

if and only if

$$M'N' - MN'' - M''N + 4uMN - 2M^2 = 0.$$

By Darboux's lemma [9, lemma 1, p.88], a nontrivial solution $y = h^*(x)$ of the 2nd order ODE

$$\frac{d^2y}{dx^2} = u^*(x;\alpha_0)y, \tag{24}$$

is given by

$$h^*(x) = \frac{1}{h(x)},$$

where $u^*(x;\alpha_0)$ and $h(x)$ are defined by (17) and (16) respectively. The transformed ODE (24) is much more simpler than the original ODE (6), if the degenerate condition is fulfilled.

In Theorem 2, the case $\text{rank}_\Lambda u(x) = 1$ is excluded. However, such potentials are already discussed precisely in [6, Corollary 14, p.424] and [7, p.977–979].

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