Proceedings of the 5th WSEAS/IASME Int. Conf. on SYSTEMS THEORY and SCIENTIFIC COMPUTATION, Malta, September 15-17, 2005 (pp226-231)

Free and Fixed End-Point Optimal Control Problems for Linear Systems with Exogenous Variables

CORNELIU BOTAN, FLORIN OSTAFI and ALEXANDRU ONEA
Automatic Control and Industrial Informatics
Technical University of Iasi
Blvd. D. Mangeron
ROMANIA

Abstract:- A comparison between the fixed end free end-point optimal control problems is performed. The problems refer to a quadratic criterion and a linear system with exogenous variables. A symmetrical algorithm for both problems is presented. This algorithm can be easier implemented by comparison with classical procedures.

Keywords: - optimal control, linear quadratic problem, free, fixed end-point

1 Introduction

A perturbed linear multivariable time invariant system is considered. The system is described by the equation

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + Bu(t) + n(t), \quad x(t_0) = x^0 \tag{1}$$

where $\tilde{x}(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control vector, n(t) is the disturbance vector and A and B are matrices of appropriate dimensions.

If x_d is the desired state vector, we can perform a translation and introduce the deviation of the state vector $x(t) = \tilde{x}(t) - x_d$ and the system equation becomes

$$\dot{x}(t) = Ax(t) + Bu(t) + w(t), \tag{2}$$

where $w(t) = Ax_d + n(t)$ (3)

is the vector of the exogenous variables (n(t) and x_d).

We shall formulate the following optimal control problems referring to the system (1):

P1. Find the optimal feedback control u(x(t)), which transfers the system (2) from the initial state $x(t_0) = x^0$ in the imposed final state $x(t_f) = 0$ and minimizes the criterion

$$J_1 = \frac{1}{2}\int_{t_0}^{t_f} [x^T(t)Q_1x(t) + u^T(t)P_1u(t)]dt \tag{4}$$

(T denotes the transposition).

P2. Find the optimal feedback control u(x(t)), which transfers the system (2) from the initial state $x(t_0) = x^0$ in the free final state $x(t_f)$ and minimizes the criterion

$$J = \frac{1}{2}x(t_f)^T Sx(t_f) + \frac{1}{2}\int_{t_0}^{t_f}\left[x^T(t)Q_1(t)x(t) + u^T(t)P_1(t)u(t)\right]dt \tag{5}$$

P1 is a fixed end-point optimal control problem and **P2** is a free end-point one.

The choice of the criterion (4) or (5) depends on the concrete application. For instance, in the motion control systems it is imposed to obtain the desired state x_d at the final moment t_f. But the problem **P2** leads to a smaller control effort and it is preferable if the final condition

$$x(t_f) = 0 \tag{6}$$

is not obligatory.

The solutions for the above formulated problems are well known [1], [2], [3] but there are some difficulties in implementation of the algorithms. The solution to the problem **P1** is usually presented as an open loop control u(t). The feedback control u(x(t)) has a complicated form and implies to compute the inverse of a time variant matrix. The problem **P2** is the most frequently problem known as a linear quadratic (LQ) optimal control problem with finite final time; the matrix of the feedback controller is time-variant and is designed based on a solution to a Riccati matriceal differential equation. This solution has to be computed in real time and this fact can generate some difficulties in implementation, augmented by the fact that the Riccati equation must be solved in inverse time, starting from a final condition.

This paper is based on some previous results of the authors [4], [5], [6], [7] and presents a simpler for implementation solution to the formulated problems. Moreover, a symmetrical approach for both problems is established. Also, a comparison between these problems is presented.

2 Usual approaches

We shall consider the same weight matrices in the above criteria ($Q_1 = Q_2 = Q$ and $P_1 = P_2 = P$), for a more relevant comparison.

Based on the Hamilton necessary conditions, the optimal control is obtained as

$$u(t) = -P^{-1}B^T\lambda(t) \tag{7}$$

where $\lambda(t) \in \mathbb{R}^n$ is the co-state vector and satisfies the equation

$$\dot{\lambda}(t) = -Qx(t) - A^T\lambda(t) \tag{8}$$

One obtain from (2), (7) and (8)

$$\begin{aligned} \dot{x}(t) &= Ax(t) - N\lambda(t) + w(t) \\ \dot{\lambda}(t) &= -Qx(t) - A^T\lambda(t) \end{aligned} \tag{9}$$

where $N = BP^{-1}B^T$.

The difference between the problem **P1** and **P2** refers to the terminal conditions:

- for the problem **P1**: $x(t_0)$ and $x(t_f) = 0$ are imposed ($\lambda(t_0)$ and $\lambda(t_f)$ are free);
- for the problem **P2**: $x(t_0)$ and

$$\lambda(t_f) = Sx(t_f) \tag{10}$$

are imposed ($x(t_f)$ and $\lambda(t_0)$ are free)

The system (9) can be written as

$$\dot{\gamma}(t) = G\gamma(t) + \mu(t), \tag{11}$$

where

$$\gamma(t) = \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} \in \mathbb{R}^{2n}, G = \begin{bmatrix} A & -N \\ -Q & -A^T \end{bmatrix} \in \mathbb{R}^{2nx2n}, \mu = \begin{bmatrix} w \\ 0 \end{bmatrix}. \tag{12}$$

The solution to the equation (11) is

$$\gamma(t) = \Gamma(t,t_0)\gamma(t_0) + \alpha(t,t_0), \tag{13}$$

where

$$\begin{aligned} \Gamma(t,t_0) &= \begin{bmatrix} \Gamma_{11}(t,t_0) & \Gamma_{12}(t,t_0) \\ \Gamma_{21}(t,t_0) & \Gamma_{22}(t,t_0) \end{bmatrix} \in \mathbb{R}^{2nx2n} \\ \Gamma_{ij}(t,t_0) &\in \mathbb{R}^{nxn} \quad i,j = 1,2 \end{aligned} \tag{14}$$

is the transition matrix for G, and

$$\alpha(t,t_0) = \begin{bmatrix} \alpha_1(t,t_0) \\ \alpha_2(t,t_0) \end{bmatrix} = \begin{bmatrix} \int_{t_0}^{t} \Gamma_{11}(t,\tau)w(\tau) \\ \int_{t_0}^{t} \Gamma_{21}(t,\tau)w(\tau) \end{bmatrix} \tag{15}$$

is the component depending on the exogenous variables. We can explicit the solution (13) for $t = t_f$

$$\begin{aligned} x(t_f) &= \Gamma_{11}(t_f,t_0)x^0 + \Gamma_{12}(t_f,t_0)\lambda^0 + \alpha_1(t_f,t_0) \\ \lambda(t_f) &= \Gamma_{21}(t_f,t_0)x^0 + \Gamma_{22}(t_f,t_0)\lambda^0 + \alpha_2(t_f,t_0) \\ \lambda^0 &= \lambda(t_0). \end{aligned} \tag{16}$$

Using (6) (for the **P1** problem), from (16) yields

$$\lambda^0 = L_1x^0 - \Gamma_{12}^{-1}(t_f,t_0)\alpha_1(t_f,t_0), \tag{17}$$

where

$$L_1 = -\Gamma_{12}^{-1}(t_f,t_0)\Gamma_{12}(t_f,t_0) \tag{18}$$

The matrix Γ_{12} is nonsingular if the system (2) is completely controllable [6].

Now, the vector $\gamma(t_0) = \begin{bmatrix} x(t_0)^T & \lambda(t_0)^T \end{bmatrix}^T$ is known and we can compute x(t) and λ(t) from (13) and then, we can express the optimal open loop control u(t) from (5).

Using the previous relations, and replacing x^0 in terms of x(t), one can obtain also the expression for the feedback control u(x(t)), but the formula is complicated and implies to compute in real time the inverse of a time variant matrix.

Similarly, using (10) and (16) we can express λ^0 in terms of x^0 for the problem **P2**, but the solution is find in this case on the other way, namely, imposing $\lambda(t) = \tilde{R}(t)x(t)$, where $\tilde{R}(t)$ is obtained as a solution to a Riccati differential matriceal equation. The difficulties which arise in this case were mentioned above.

3 Main results

A significant simplification is obtained if we perform a change of variables:

$$\gamma(t) = T\rho(t) \qquad (19)$$

with

$$T = \begin{bmatrix} I_n & 0 \\ R & I_n \end{bmatrix} \text{ and } T^{-1} = \begin{bmatrix} I_n & 0 \\ -R & I_n \end{bmatrix} \qquad (20)$$

where I_n is the nxn identity matrix and R is a symmetrical nxn matrix.
According to (20), the new vector is

$$\rho(t) = \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ \lambda(t) - Rx(t) \end{bmatrix} \qquad (21)$$

and the corresponding differential equation is

$$\dot{\rho}(t) = H\rho(t) + T^{-1}\mu(t), \qquad (22)$$

with

$$H = T^{-1}GT \qquad (23)$$

From (12) and (20) we obtain

$$H = \begin{bmatrix} A - NR & -N \\ RNR - RA - A^T R - Q & -A^T + RN \end{bmatrix} \qquad (24)$$

If we choose R so that

$$RNR - RA - A^T R - Q = 0 \qquad (25)$$

and denote

$$F = A - NR, \qquad (26)$$

the matrix H becomes

$$H = \begin{bmatrix} F & -N \\ 0 & -F^T \end{bmatrix} \qquad (27)$$

Note that (25) is the Riccati matriceal algebraic equation which appears in the LQ problem with infinite final time.

Let be $\Omega(t,t_c)$ the transition matrix for H. Based on the fact that $\dot{\Omega}(t,t_c) = H\Omega(t,t_c)$, one obtain [4]

$$\Omega(t,t_c) = \begin{bmatrix} \Psi(t,t_c) & \Omega_{12}(t,t_c) \\ 0 & \phi(t,t_c) \end{bmatrix} \qquad (28)$$

where $\Psi(.)$ and $\phi(.)$ are the transition matrices for F and $-F^T$, respectively, where t_c denotes the terminal time (t_0 or t_f) and

$$\Omega_{12}(t,t_c) = \int_t^{t_c} \Psi(t,\tau) N \phi(\tau,t_c) d\tau \qquad (29)$$

The solution to the equation (22) is

$$\rho(t) = \Omega(t,t_c)\rho(t_c) + \beta(t,t_c), \qquad (30)$$

where

$$\beta(t,t_c) = \begin{bmatrix} \beta_1(t,t_c) \\ \beta_1(t,t_c) \end{bmatrix} = \begin{bmatrix} \int_{t_c}^{t} [\Psi(t,\tau) - \Omega_{12}(t,\tau)R] w(\tau) d\tau \\ \int_{t_c}^{t} -\Phi(t,\tau) R w(\tau) d\tau \end{bmatrix}, \qquad (31)$$

is the component depending on the exogenous variables. The solution (30) can be expressed as

$$x(t) = \Psi(t,t_c)x(t_c) + \Omega_{12}(t,t_c)v(t_c) + \beta_1(t,t_c) \qquad (32)$$

$$v(t) = \Phi(t,t_c)v(t_c) + \beta_2(t,t_c) \qquad (33)$$

For the problem P1, for $t_c = t_0$, one obtain from (17) and (21)

$$v(t_0) = \lambda^0 - Rx^0 = (L_1 - R)x^0 - \Gamma_{12}^{-1}(t_f,t_0)\alpha_1(t_f,t_0) \qquad (34)$$

with L_1 given by (18).
Thus, the solution for v(t) can be expressed from (33) in terms of x^0 and of exogenous vector

$$v(t) = \Phi(t,t_0)v(t_0) + \beta_2(t,t_0), \qquad (35)$$

with $v(t_0)$ given by (34) and $\beta_2(t,t_0)$ given by (31). The last term in (34) can be easier computed if we express the transition matrix $\Gamma(.)$ in terms of the transition matrix $\Omega(.)$. For this purpose, one obtain from (23)

$$\Gamma(.) = T\Omega(.)T^{-1} \qquad (36)$$

and therefore, using (20) and (28), it results

$$\Gamma_{11}(.) = \Psi(.) - \Omega_{12}(.)R \quad \text{and} \quad \Gamma_{12}(.) = \Omega_{12}(.). \qquad (37)$$

For the P2 problem we use the final condition (10) and from (21) for $t = t_f$, it results

$$v(t_f) = (S - R)x(t_f). \qquad (38)$$

We obtain from (32) for $t = t_0$ and $t_c = t_f$ and from (38)

$$x^0 = Mx(t_f) + \beta_1(t_0, t_f), \qquad (39)$$

where

$$M = \Psi(t_0, t_f) + \Omega_{12}(t_0, t_f)(S - R). \qquad (40)$$

Now, we can write from (38) and (39)

$$v(t_f) = (S - R)M^{-1}[x^0 - \beta_1(t_0, t_f)] \qquad (41)$$

(one can prove that M is a nonsingular matrix).
Using (33) for $t = t_0$ and $t_c = t_f$ and (39), we obtain

$$v(t_0) = V[x^0 - \beta_1(t_0, t_f)] + \beta_2(t_0, t_f) \qquad (42)$$

with

$$V = \Phi(t_0, t_f)(S - R)M^{-1} \qquad (43)$$

and the vector v(t) is given by the same expression (35) as for the **P1** problem, but now $v(t_0)$ is computed with (42).

For both problems, the optimal control vector can be computed from (7) and (21) in the form

$$u(t) = u_f(t) + u_s(t) \qquad (45)$$

where

$$u_f(t) = -P^{-1}B^T Rx(t) \qquad (46)$$

is the feedback component and

$$u_s(t) = -P^{-1}B^T v(t) \qquad (47)$$

is a supplementary component, depending on the vector v(t), given by (35).

Remark 1: The optimal control u(t) contains only the feedback component $u_f(t)$ for the problem LQ problem with infinite final time. In the cases of the **P1** and **P2** problems appears a supplementary component $u_s(t)$ given by (35). This component depends on the initial state x^0 and on the exogenous variables x_d and n(t). The difference between the problems **P1** and **P2** consists in the formulae for the initial value $v(t_0)$ of the vector v(t) ((34) for **P1** and (42) for **P2**).

Remark 2: The computation of optimal control u(t) implies the knowledge of the vectors $\beta_1(t_0,t_f)$ and $\beta_2(t_0,t_f)$. This supposes that the exogenous variables are beforehand known on the optimization interval $[t_0,t_f]$ and this implies to know the value of the disturbance n(t) on this interval. Only the knowledge of the shape of n(t) is sufficient (for instance n(t)=constant) if the disturbance torque is measured or estimated at the initial moment. A disturbance observer can be introduced in the controller structure on this purpose.

Remark 3: The above formulae are quite complicated, but the most part of the computing is performed off-line, in the design stage of the controller. The real-time computing implies only the computing of the optimal control u(t) and this imposes to establish a usual feedback component $u_f(t)$ and a supplementary component $u_s(t)$. This last component contains in the both cases only two time variant elements – the matrix $\Phi(t, t_0)$ and the vector $\beta_2(t,t_f)$. These variant elements can be recurrently computed and thus, the supplementary component can be easy computed and the global computing effort for u(t) is not too complicated by comparison with the case of the usual state feedback control.

Remark 4: The optimal control is ensured for the both problems on the interval $[t_0,t_f]$. In many cases it is desired to maintain the desired values of the variables for $t > t_f$. It is necessary in this case to change the control law, for instance to introduce a usual linear feedback control.

4 Simulation results

Some simulation tests were performed for both problems **P1** and **P2**. The Fig.1 and 2 show the behaviour of the optimal system in the case of the LQ problem with fixed end-point and in the case of the LQ problem with free end-point, respectively. The presented results are for the following data (the same as in [7], which refers to similar problems but without exogenous variables).

$$A = \begin{bmatrix} 0 & 0.1 & 0 \\ 0 & 0 & 20 \\ 0 & -3.5 & -19.4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 6.25 \end{bmatrix}, \quad n = \begin{bmatrix} 0 \\ -28 \\ 0 \end{bmatrix}$$

(the adopted matrices correspond for instance to a positioning system).

$$Q = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3.1 \end{bmatrix}, \quad S = \begin{bmatrix} 60 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$P = p = 1,\ t_0 = 0s,\ t_f = 1s$.

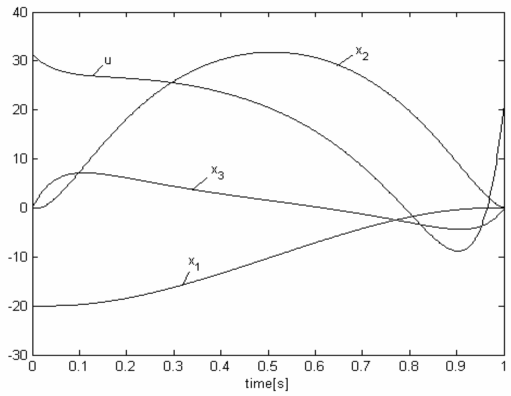

Fig. 1

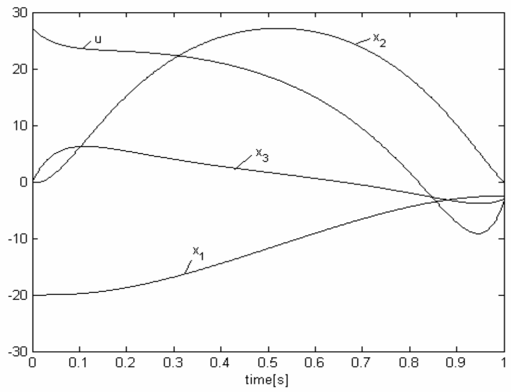

Fig. 2

One can remark a good behaviour of the optimal system in both cases and that $x(t_f)=0$ can be achieved only for the **P1** problem, but the control variable u(t) is bigger in this case, especially in the initial and final period. This result is expected because the system is forced in this case to reach the imposed final state $x(t_f) = 0$.

The effect of an imprecise estimation is presented in the Fig. 3 and 4 for the two problems (an error of 35% is supposed). One can remark that especially the final values of the state variables are affected.

Finally, the Fig. 5 and 6 show the system behaviour for the two problem if the control law is changed for $t > t_f$, as it is indicated in the Remark 4. The modified control law was chosen so that only the final values for the variables x_1 and x_2 to be zero.

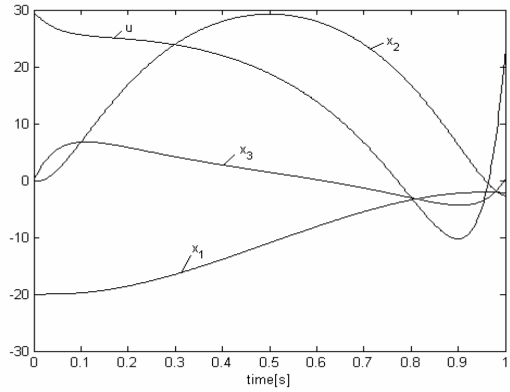

Fig. 3

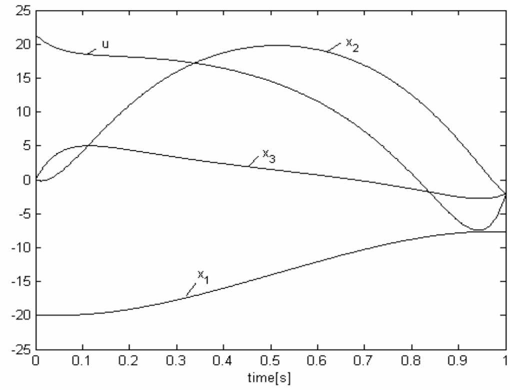

Fig.4

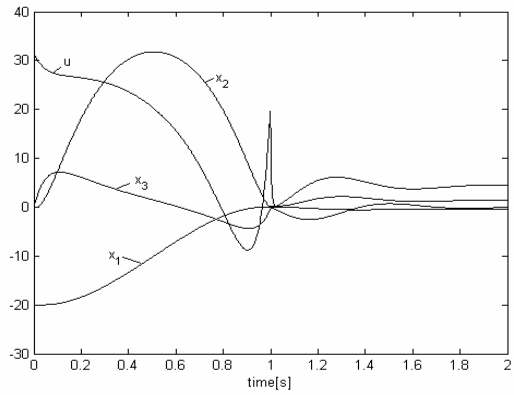

Fig.5

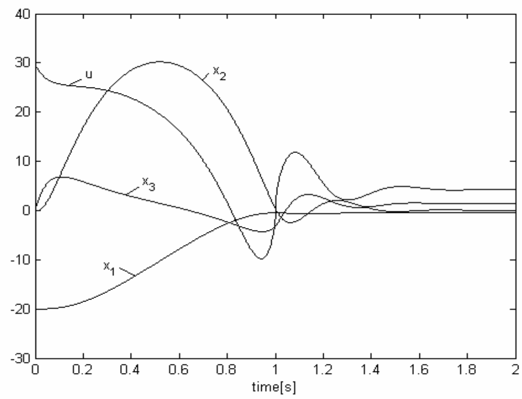

Fig. 6

5 Conclusions

- A comparison between LQ optimal problem with fixed end-point and free end-point is performed. The effect of the exogenous variables (desired state vector and disturbance) is studied.
- The algorithms indicated in the paper for the both problems have advantages by comparison with classical procedures and lead to a significant decrease of the computing time.
- Unlike the usual approaches, which are different for the both problems, the proposed method leads to a similar solution for the both problems: the optimal control contains similar feedback and supplementary components; the difference is between the last components, which involve different initial values for a vector.
- The supplementary component contains two terms in both cases: one depending on the initial state and one depending on the exogenous variables.

References

[1] M. Athans, P.L. Falb, *Optimal Control*, Mc Graw Hill, New York, 1966.

[2] B.D.O. Anderson, J.B. Moore, *Optimal Linear Quadratic Methods*, Prentice Hall, Englewood Cliffs, New Jersey, 1990.

[3] L. Lapidus, R. Luus, *Optimal Control of Engineering Processes*, Blaisdell Publishing Company, Waltham Massachusetts, 1967.

[4] C. Botan, On the Solution of the Riccati Differential Matrix Equation, *Symposium on Computational System Analyses*, Elsever Science Publishers, Berlin, 1992, pp. 141-146.

[5] C. Botan, A. Onea, A fixed End-Point Problem for an Electrical Drive System, *Proceedings of the IEEE International Symposium on Industrial Electronics, ISIE'99*, Bled, Slovenia, Vol. 3, 1999, pp. 1345-1349.

[6] C. Botan, F. Ostafi, A. Onea, A solution to the fixed end-point linear quadratic optimal problem, *Advances in Automatic Control (M. Voicu ed.)*, Kluwer Academic Publishers, Boston, 2003, pp. 9-20.

[7] C. Botan, F. Ostafi, A. Onea, Free and Fixed End-Point Optimal Control Problems for Linear Systems, *WSES Transactions on Systems,* Issue 3, Volume 3, May 2004, ISSN 1109-2777, pp. 1109- 1113.