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System Representation by a set of Low Dimensional Models

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1. Introduction

The paper combines the approaches known in signal processing area with the approaches typical for neural networks. In the signal processing community the signal parameters as mean values, correlation matrix, covariance matrix, regression parameters vector, etc. are estimated through measured data. The knowledge of such parameters results in the identification of the optimal LTI (linear time invariant) model describing the main system property [1]. The model with estimated parameters could be used for time interpolation, filtering, extrapolation, etc.

2. Sylvester's theorems

The decomposition methodology described in this chapter is based on the application of Sylvester's theorems on system transition matrix **A** of *m*-dimensional linear time invariant system described by state-space model:

$$\begin{aligned} x_{n+1} &= \mathbf{A}\bullet x_n + \mathbf{B}\bullet u_n \\ y_n &= \mathbf{C}\bullet x_n + \mathbf{D}\bullet u_n \end{aligned} \qquad (1)$$

where x_n, u_n, y_n are *m*-dimensional state, input and output vectors in time interval *n* and **A**,**B**,**C**,**D** are state-space $m \times m$ matrices.

Theorem 1: Sylvester theorem for distinct eigenvalues

If **A** is square system transition matrix (1) and if λ_i represents one of the *n* distinct eigenvalues of **A**, and if P(**A**) is any polynomial of the matrix **A**, then

$$P(\mathbf{A}) = \sum_{i=1}^{n} \frac{P(\lambda_i)\bullet \mathrm{Adj}(\mathbf{A}-\lambda_i \bullet \mathbf{I})}{\prod_{j\neq i}^{n}(\lambda_j-\lambda_i)} = \sum_{i=1}^{n} P(\lambda_i)\bullet \prod_{j\neq i}^{n} \frac{(\mathbf{A}-\lambda_j\bullet\mathbf{I})}{(\lambda_i-\lambda_j)} \qquad (2)$$

where Adj(**A**) means the adjoin matrix that is formed by replacing each element of matrix A by its cofactor and then taking the transpose.

Theorem 2: Sylvester theorem for repeated eigenvalues

If **A** is square system transition matrix (1) and if λ_i represents an eigenvalue of **A** repeated s_i times, and if *k* is number of all distinct eigenvalues λ_j, and if P(**A**) is any polynomial of the matrix **A**, then

$$P(\mathbf{A}) = \sum_{\substack{i=1 \\ (\text{all distinct} \\ \text{eigenvalues})}}^{k} \frac{(-1)}{(s_i-1)!}\left[\frac{d^{s_i-1}}{d\lambda^{s_i-1}}\left(\frac{P(\lambda)\bullet \mathrm{Adj}(\mathbf{A}-\lambda\bullet\mathbf{I})}{\prod_{j\neq i}^{k}(\lambda-\lambda_j)}\right)\right]_{\lambda=\lambda_i} \qquad (3)$$

where Adj(**A**) means the adjoin matrix and if all eigenvalues are equal, then $\prod_{j\neq i}^{k}(\lambda-\lambda_j)=1$.

Definition 1:

With respect to equation (2) and with assumption of distinct eigenvalues the special matrixes $Z_1, Z_2, ..., Z_n$ could be defined for matrix **A**:

$$Z_i = \prod_{j\neq i}^{n}\frac{(\mathbf{A}-\lambda_j\bullet\mathbf{I})}{(\lambda_i-\lambda_j)} \qquad (4),$$ with following properties:

$$\begin{aligned} &Z_i\bullet Z_j = 0 \quad \text{for } i \neq j, \\ &Z_i\bullet Z_j = Z_i \quad \text{for } i = j, \\ &\sum_{i=1}^{n} Z_i = 1. \end{aligned} \qquad (5)$$

Proof: The proof of theorem 1 and 2 is done in [4] where the proof of matrix components for repeated eigenvalues is also presented.

3. Approximation of derivatives

In equation (3) the derivatives are necessary to be solved. In following theorem the form of derivatives approximation is presented together with the approximation error.

Theorem 3: Approximation of derivatives

Let f(*x*) is any function of *x* and $\frac{df(x)}{dx}, \frac{d^2f(x)}{dx^2}, \frac{d^3f(x)}{dx^3}....$ first, second, third, etc. derivatives of function f(*x*), then the approximate derivatives could be expressed

$$\begin{aligned} \frac{df(x)}{dx} &\approx \frac{1}{2h}\big(f(x+h)-f(x-h)\big), \\ \frac{d^2f(x)}{dx^2} &\approx \frac{1}{2h^2}\big(f(x+h)-2f(x)+f(x-h)\big), \\ \frac{d^3f(x)}{dx^3} &\approx \frac{1}{2h^3}\big(f(x+2h)-2f(x+h)+2f(x-h)-f(x-2h)\big), \\ &\text{etc.} \end{aligned} \qquad (6)$$

where for all derivatives the approximation error is proportional to $.\left(h^2\right)$.

For better precision of approximation other approximate forms exist, e.g. for approximation error proportional to $.\left(h^4\right)$ the following equations could be described:

$$\frac{df(x)}{dx}\ \frac{1}{12h}\left(-f(x+2h)+8f(x+h)-8f(x-h)+f(x-2h)\right),$$
$$\frac{d^2f(x)}{dx^2}\ \frac{1}{12h^2}\left(-f(x+2h)+16f(x+h)-30f(x)+16f(x-h)-f(x-2h)\right),$$
etc.

(7)

Proof: The proof is given in [4].

Theorem 4: Complex-step approximation of derivatives

Let f(*x*) is any function of *x* and $\frac{df(x)}{dx}$ is first derivative of function f(*x*), then the approximate derivatives with complex-step approximation could be expressed:

$$\frac{df(x)}{dx}=\lim_{h.\ 0}\frac{\mathrm{Im}\left[f(x+j\bullet h)\right]}{h}\quad\frac{\mathrm{Im}\left[f(x+j\bullet h)\right]}{h} \tag{8}$$

where Im[.] means imaginary part. Then the approximation error is proportional to $.\left(h^2\right)$.

Proof: Let us express $z=x+j\bullet y$ and $f(z)=u+j\bullet v$ together with Cauchy-Riemann equations [6]:

$$\frac{.u}{.x}=\frac{.v}{.y}\quad(9),\qquad\qquad \frac{.u}{.y}=-\frac{.v}{.x}\quad(10)$$

If *f* is analytic function (satisfy the (9) and (10)) than we can use and rewrite (9) as follows:

$$\frac{.u}{.x}=\lim_{h.\ 0}\frac{v(x+j\bullet(y+h))-v(x+j\bullet y)}{h} \tag{11}$$

Since function f is real function of a real variable than $y=0$, $u(x)=f(x)$ and $v(x)=0$. Equation (11) can be than rewritten to (8). In order to determine the error involved in this approximation, the derivation based on Taylor series expansion is used (with pure imaginary step equal to $j\bullet h$):

$$f(x+j\bullet h)=f(x)+j\bullet h\bullet\frac{df(x)}{dx}-h^2\bullet\frac{1}{2!}\bullet\frac{d^2f(x)}{dx^2}+j\bullet h^3\bullet\frac{1}{3!}\bullet\frac{d^3f(x)}{dx^3}-\ldots\ldots \tag{12}$$

Taking imaginary part of both sides of equation (12) and dividing this equation by *h* yields:

$$\frac{df(x)}{dx}=\frac{\mathrm{Im}\left[f(x+j\bullet h)\right]}{h}+h^2\bullet\frac{1}{3!}\bullet\frac{d^3f(x)}{dx^3}+\ldots.. \tag{13}$$

Hence the approximation is $.\left(h^2\right)$. The higher order derivatives express in Theorem 3 could be transferred into complex-step method by using approximation of Cauchy's Integral Formula in general form (Γ is a simple closed positively oriented contour that encloses *z*):

$$\frac{d^nf(z)}{dz^n}=\frac{n!}{2\bullet\pi\bullet j}\int_{\Gamma}\frac{f(.\,)}{(.\,-z)^{n+1}}d.\quad\frac{n!}{m\bullet r}\bullet\sum_{i=0}^{m-1}\frac{f\left(z+r\bullet e^{j\bullet\frac{2\bullet\pi\bullet i}{m}}\right)}{e^{j\bullet\frac{2\bullet\pi\bullet i\bullet n}{m}}} \tag{14}$$

where *m* is the number of points used in the integration. the approximate of derivative of order $n=0,1,\ldots.,m-1$ could be fond by using (14). From complex variable theory, for a real function of the real variable that is analytic holds: $f(x+j\bullet y)=u+j\bullet v\Rightarrow f(x-j\bullet y)=u-j\bullet v$ (15).

4. Approximation of derivatives in Sylvester theorem for repeated eigenvalues (transformed eigenvalues method)

The approximation of derivatives (6), (7) could be applied to Sylvester theorem for repeated eigenvalues (3) and the following theorem could be defined.

Theorem 5: Approximation of P(**A**) with repeated eigenvalues of matrix **A**

If matrix **A** has *k* all distinct eigenvalues (3) where *d* eigenvalues $._d$ are repeated s_d times and *ch* eigenvalues $._{ch}$ are poorly distinct, then the polynomial function P(**A**) could be approximated by set of *ch* distinct and by set of *t* **transformed distinct eignevalues** $._t$ $(._1-q_1h,..,._1+q_1h,..,._d-q_dh,\ldots.,._d+q_dh\quad)$ with error proportional at least to $O(h^2)$ as follows:

$$P(\mathbf{A})=\sum_{i=1}^{ch}P(._i)\bullet\prod_{\propto.\,i}^{k}\frac{(\mathbf{A}-.\,_{\propto}\bullet\mathbf{I})}{(._i-._{\propto})}+\sum_{f=1}^{d}\sum_{.=-q_f}^{q_f}k_{f,.}\bullet P(._f+.\bullet h)\bullet\prod_{\propto.\,f}^{k}\frac{(\mathbf{A}-.\,_{\propto}\bullet\mathbf{I})}{(._f+.\bullet h-._{\propto})}=$$
$$=\sum_{i=1}^{ch}P(._i)\bullet Z_i+\sum_{f=1}^{d}\sum_{.=-q_f}^{q_f}k_{f,.}\bullet P(._f+.\bullet h)\bullet Z_{f,.} \tag{16}$$

where q_f depends on selected approximate form (6), (7), (8) and on the multiplicity of repeated eigenvalue $._f$, $k_{f,.}$ are weight constants of approximation form (6), (7), (8) $Z_{f,.}=\prod_{\propto.\,f}^{k}\frac{(\mathbf{A}-.\,_{\propto}\bullet\mathbf{I})}{(._f+.\bullet h-._{\propto})}$ are transformed component matrices assigned to transformed distinct

Tab.1: Modified roots in Lagrange polynomial [4]

modified roots/ repeated roots	(a-2h)	(a-h)	a	(a+h)	(a+2h)	**approx. error**
$(x+a)^2$	0	$(x+(a-h))$	0	$(x+(a+h))$	0	h^2
$(x+a)^3$	0	$(x+(a-h))$	$(x+a)$	$(x+(a+h))$	0	$h^2(x+a)$
$(x+a)^4$	$(x+(a-2h))$	$(x+(a-h))$	0	$(x+(a+h))$	$(x+(a+2h))$	$4h^4 - -5h^2(x+a)^2$
$(x+a)^5$	$(x+(a-2h))$	$(x+(a-h))$	$(x+a)$	$(x+(a+h))$	$(x+(a+2h))$	$4h^4(x+a)- -5h^2(x+a)^3$

eigenvalues and *h* is selected small approximation parameter.

Proof:

If the Sylvester theorem for repeated eigenvalues (3) is used and if the approximation according to equation (6), (7) or (8) is applied then the part of equation (3) assigned to repeated eigenvalue $\cdot_i$ could be approximated with error proportional at least to $O(h^2)$ or more (depends on used approximate form, e.g. forms (6), (7) or (8)) as follows:

$$\frac{(-1)}{(s_i-1)!}\left[\frac{d^{s_i-1}}{d\cdot^{s_i-1}}\left(\frac{P(\cdot)\bullet Adj(\mathbf{A}-\cdot\bullet\mathbf{I})}{\prod_{j\ne i}^{k}(\cdot-\cdot_j)}\right)\right]_{\cdot=\cdot_i} = \frac{(-1)}{(s_i-1)!}\bullet\left[\frac{d^{s_i-1}}{d\cdot^{s_i-1}}\bullet P(\cdot)\bullet Z(\cdot)\right]_{\cdot=\cdot_i} =$$
$$= k_{-q_i}\bullet P(\cdot_i-q_i\bullet h)\bullet Z_{i,-q_i}+..+k_0\bullet P(\cdot_i)\bullet Z_{i,0}+..+k_{q_i}\bullet P(\cdot_i+q_i\bullet h)\bullet Z_{i,q_i} \quad (17)$$

where $\cdot_i-q_i\bullet h,..,\cdot_i,...,\cdot_i+q_i\bullet h$ are transformed eigenvalues assigned to s_i times repeated eigenvalue $\cdot_i$ and $Z_{i,-q_i},..,Z_{i,0},...,Z_{i,q_i}$ are transformed component matrices assigned to transformed eigenvalues $\cdot_i-q_i\bullet h,..,\cdot_i,...,\cdot_i+q_i\bullet h$.

5. Approximation based on modified repeated eigenvalues (modified eigenvalues method)

The Sylvester' theorem came out from Lagrange interpolation polynomial with distinct polynomial roots. The repeated roots in Lagrange polynomial could be modified and approximated by distinct roots according to following table with defined approximation error (it is easy to extend the table for higher root multiplicity):

In case the parameter *h* is small enough the approximation error could be also small because of high power of *h*. This methodology could be applied to Sylvester's theorem and the **modified eigenvalues** could be used instead of repeated ones and then the decomposition could be done with help of equation (2). As an example following table describes the multiplicity of original repeated eigenvalue and its modification to distinct ones:

Tab. 2 Original repeated and modified distinct eigenvalues

Multiplicity of original repeated eigenvalues	**Modified distinct eigenvalues**
$\cdot^2$	$\cdot-h,\ \cdot+h$
$\cdot^3$	$\cdot-h,\ \cdot,\ \cdot+h$
$\cdot^4$	$\cdot-2h,\ \cdot-h,\ \cdot+h,\ \cdot+2h$
$\cdot^5$	$\cdot-2h,\ \cdot-h,\ \cdot,\ \cdot+h,\ \cdot+2h$
etc.	etc.

The LTI dynamical system decomposition is based on the application of Sylvester's theorems or its approximation to state-space matrix **A** (1) and by using the property of components matrices to calculate the transformed input, output and state-space vectors of one-dimensional models.

6. The LTI system decomposition with distinct eigenvalues of transition matrix

The LTI system decomposition with distinct eigenvalues could be done with help of following fundamental decomposition theorem.

Theorem 6: Fundamental decomposition of LTI dynamical systems with distinct eigenvalues of transition matrix **A**

Dynamical *m*-dimensional LTI dynamical system described by state-space model (1) with transition matrix **A** with distinct eigenvales could be decomposed into m^2 one-dimensional LTI models where the component matrices assigned to transition matrix **A** are used as transformation matrices of state, input and output vectors (filter banks).

Proof:

The *m*-dimensional state-space model (1) with square $m \cdot m$ matrices A, B, C, D[1] could be decomposed with help of components matrices $Z_1, Z_2, ..., Z_m$ into following form

$$\begin{aligned} & Z_1 x_{n+1} + Z_2 x_{n+1} + ... + Z_m x_{n+1} = \\ & = \cdot_1 Z_1 x_n + \cdot_2 Z_2 x_n + ... + \cdot_m Z_m x_n + Z_1 \bullet B \bullet u_n + Z_2 \bullet B \bullet u_n + .. + Z_m \bullet B \bullet u_n \\ & y_{1,n} + y_{2,n} + .. + y_{m,n} = Z_1 \bullet y_n + Z_2 \bullet y_n + ... + Z_m \bullet y_n = \\ & = Z_1 \bullet C \bullet x_n + Z_2 \bullet C \bullet x_n + .. + Z_m \bullet C \bullet x_n + Z_1 \bullet D \bullet u_n + ... + Z_m \bullet D \bullet u_n \end{aligned} \tag{18}$$

where form P(A)=A in equation (2) was used and the matrix components property $\sum_{i=1}^{m} Z_i = 1$ (5) was taking into account. By multiplying the equations (18) by component matrix Z_j and by taking into account the property $Z_i \bullet Z_j = 0 \;\; \text{for}\; i \cdot j$ and $Z_i \bullet Z_j = Z_i \;\; \text{for}\; i = j$ (5) the LTI dynamical system could be decomposed into *m* following sub-systems:

$$\begin{aligned} & Z_j \bullet x_{n+1} = \cdot_j \bullet Z_j \bullet x_n + Z_j \bullet B \bullet u_n \\ & y_{j,n} = Z_j \bullet y_n = Z_j \bullet C \bullet x_n + Z_j \bullet D \bullet u_n \end{aligned} \tag{19}$$

with transition value equal to $\cdot_j$. The transformed *m*-dimensional state vectors $Z_j \bullet x_n \;\; j \cdot \{1,2,...,m\}$ were obtained through filtering of state vector by component matrices (component matrices play the role of filter banks). Each component of transformed state vector $Z_j \bullet x_n$ could be easily described as a first-order dynamical model because the transition value $\cdot_j$ is common for all *m* transformed states $Z_j \bullet x_n$.

7. Identification methods of low dimensional models

The identification task yields to estimation of *s*-dimensional vector unknown parameters:

$$\vec{w} = [a_1, a_2, ..., a_m, b_1, b_2, ..., b_m, p_1, p_2, ... p_m, \cdot_1, \cdot_2, ..., \cdot_m] \tag{20}$$

based on the knowledge of data vector:

$$\vec{x}_i = [x_{1,i}, x_{2,i}, ..., x_{m,i}, x_{1,i-1}, x_{2,i-1}, ..., x_{m,i-1}] \tag{21}$$

where the transformed state-vector fulfill the conditions on *m* one-dimensional models:

$$\begin{bmatrix} a_1 & b_1 & .. & p_1 \\ a_2 & b_2 & .. & p_2 \\ .. & .. & .. & .. \\ a_m & b_m & .. & p_m \end{bmatrix} \bullet \begin{bmatrix} x_{1,i} \\ x_{2,i} \\ .. \\ x_{m,i} \end{bmatrix} = \begin{bmatrix} \cdot_1 & 0 & .. & 0 \\ 0 & \cdot_2 & .. & .. \\ .. & .. & .. & 0 \\ 0 & .. & 0 & \cdot_m \end{bmatrix} \bullet \begin{bmatrix} a_1 & b_1 & .. & p_1 \\ a_2 & b_2 & .. & p_2 \\ .. & .. & .. & .. \\ a_m & b_m & .. & p_m \end{bmatrix} \bullet \begin{bmatrix} x_{1,i-1} \\ x_{2,i-1} \\ .. \\ x_{m,i-1} \end{bmatrix} \tag{22}$$

The parameters must be estimated under the pre-defined composition rule:

$$\begin{bmatrix} x_{1,i} \\ .. \\ x_{m,i} \end{bmatrix} = \begin{bmatrix} r_{1,1} & .. & r_{1,m} \\ .. & .. & .. \\ r_{m,1} & .. & r_{m,m} \end{bmatrix} \bullet \begin{bmatrix} a_1 & .. & p_1 \\ .. & .. & .. \\ a_m & .. & p_m \end{bmatrix} \bullet \begin{bmatrix} x_{1,i} \\ .. \\ x_{m,i} \end{bmatrix} \tag{23}$$

where parameters $r_{1,1}, ..., r_{m,1}, ..., r_{m,m}$ are given beforehand.

The (22) could be rewritten into following form:

$$\begin{bmatrix} f_1(\vec{x}_i, \vec{w}) \\ f_2(\vec{x}_i, \vec{w}) \\ .. \\ f_m(\vec{x}_i, \vec{w}) \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & .. & p_1 \\ a_2 & b_2 & .. & p_2 \\ .. & .. & .. & .. \\ a_m & b_m & .. & p_m \end{bmatrix} \bullet \begin{bmatrix} x_{1,i} \\ x_{2,i} \\ .. \\ x_{m,i} \end{bmatrix} - \begin{bmatrix} \cdot_1 & 0 & .. & 0 \\ 0 & \cdot_2 & .. & .. \\ .. & .. & .. & 0 \\ 0 & .. & 0 & \cdot_m \end{bmatrix} \bullet \begin{bmatrix} a_1 & b_1 & .. & p_1 \\ a_2 & b_2 & .. & p_2 \\ .. & .. & .. & .. \\ a_m & b_m & .. & p_m \end{bmatrix} \bullet \begin{bmatrix} x_{1,i-1} \\ x_{2,i-1} \\ .. \\ x_{m,i-1} \end{bmatrix} \tag{24}$$

with the additional functions representing composition rule:

$$\begin{bmatrix} f_{m+1}(\vec{x}_i, \vec{w}) \\ .. \\ f_n(\vec{x}_i, \vec{w}) \end{bmatrix} = \begin{bmatrix} x_{1,i} \\ .. \\ x_{m,i} \end{bmatrix} - \begin{bmatrix} r_{1,1} & .. & r_{1,m} \\ .. & .. & .. \\ r_{m,1} & .. & r_{m,m} \end{bmatrix} \bullet \begin{bmatrix} a_1 & .. & p_1 \\ .. & .. & .. \\ a_m & .. & p_m \end{bmatrix} \bullet \begin{bmatrix} x_{1,i} \\ .. \\ x_{m,i} \end{bmatrix} \tag{25}$$

The (24) and (25) are set of *n* non-linear functions that could be linearized by matrix Taylor series as follows (from (24) and (25) it arises that the functions $f_1(\vec{x}, \vec{w}_i), ..., f_n(\vec{x}, \vec{w}_i)$ must converge to zero vector):

$$\begin{bmatrix} f_1(\vec{x}_i, \vec{w}_i) \\ f_2(\vec{x}_i, \vec{w}_i) \\ .. \\ f_n(\vec{x}_i, \vec{w}_i) \end{bmatrix} = \begin{bmatrix} f_1(\vec{x}_i, \vec{w}_{i-1}) \\ f_2(\vec{x}_i, \vec{w}_{i-1}) \\ .. \\ f_n(\vec{x}_i, \vec{w}_{i-1}) \end{bmatrix} + \begin{bmatrix} \frac{\cdot f_1(\vec{x}_i, \vec{w})}{\cdot w_1} & \frac{\cdot f_1(\vec{x}_i, \vec{w})}{\cdot w_2} & .. & \frac{\cdot f_1(\vec{x}_i, \vec{w})}{\cdot w_s} \\ .. & .. & .. & .. \\ \frac{\cdot f_n(\vec{x}_i, \vec{w})}{\cdot w_1} & \frac{\cdot f_n(\vec{x}_i, \vec{w})}{\cdot w_2} & .. & \frac{\cdot f_n(\vec{x}_i, \vec{w})}{\cdot w_s} \end{bmatrix}_{\vec{w} = \vec{w}_{i-1}} \bullet \left(\begin{bmatrix} w_{1,i} \\ w_{2,i} \\ .. \\ w_{s,i} \end{bmatrix} - \begin{bmatrix} w_{1,i-1} \\ w_{2,i-1} \\ .. \\ w_{s,i-1} \end{bmatrix} \right) \begin{bmatrix} 0 \\ 0 \\ .. \\ 0 \end{bmatrix} \tag{26}$$

In next part we mark the vector of unknown parameters in *i*-time step as $\vec{w}_i$:

$$\vec{w}_i = \begin{bmatrix} w_{1,i} & w_{2,i} & ... & w_{s,i} \end{bmatrix}^T \tag{27}$$

The measurement vector will be marked as $\vec{z}_{i-1}$ and transition matrix between measurement and parameter vector as D_{i-1}. The vector $\vec{z}_{i-1}$ and matrix D_{i-1} are computed from known data vector $\vec{x}_i$ and last estimate of parameters $\vec{w}_{i-1}$ as follows:

$$\vec{z}_{i-1} = \begin{bmatrix} \frac{\cdot f_1(\vec{x}_i, \vec{w})}{\cdot w_1} & \frac{\cdot f_1(\vec{x}_i, \vec{w})}{\cdot w_2} & .. & \frac{\cdot f_1(\vec{x}_i, \vec{w})}{\cdot w_s} \\ .. & .. & .. & .. \\ \frac{\cdot f_n(\vec{x}_i, \vec{w})}{\cdot w_1} & \frac{\cdot f_n(\vec{x}_i, \vec{w})}{\cdot w_2} & .. & \frac{\cdot f_n(\vec{x}_i, \vec{w})}{\cdot w_s} \end{bmatrix}_{\vec{w} = \vec{w}_{i-1}} \bullet \begin{bmatrix} w_{1,i-1} \\ w_{2,i-1} \\ .. \\ w_{s,i-1} \end{bmatrix} - \begin{bmatrix} f_1(\vec{x}_i, \vec{w}_{i-1}) \\ f_2(\vec{x}_i, \vec{w}_{i-1}) \\ .. \\ f_n(\vec{x}_i, \vec{w}_{i-1}) \end{bmatrix} \tag{28}$$

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[1] for non-square matrices B,C,D, the zero elements could be completed as well as in vectors u_n, y_n to achieve the form (4.1)

$$D_{i-1} = \begin{bmatrix} \frac{\partial f_1(\vec{x}_i,\vec{w})}{\partial w_1} & \frac{\partial f_1(\vec{x}_i,\vec{w})}{\partial w_2} & .. & \frac{\partial f_1(\vec{x}_i,\vec{w})}{\partial w_s} \\ .. & .. & .. & .. \\ & & .. & \\ \frac{\partial f_n(\vec{x}_i,\vec{w})}{\partial w_1} & \frac{\partial f_n(\vec{x}_i,\vec{w})}{\partial w_2} & .. & \frac{\partial f_n(\vec{x}_i,\vec{w})}{\partial w_s} \end{bmatrix}_{\vec{w}=\vec{w}_{i-1}} \quad (29)$$

Based on representation (28) and (29) the equation for extended Kalman filter [14] could be written:

$$\vec{z}_{i-1} = D_{i-1} \bullet \vec{w}_i + \vec{e}_i \quad (30)$$

where noise vector $\vec{e}_i$ is supposed to be Gaussian with zero mean and covariance matrix Q:

$$\mathrm{cov}[\vec{e}_i,\vec{e}_j] = 0 \quad i \neq j, \quad \mathrm{cov}[\vec{e}_i,\vec{e}_j] = Q \quad i = j \quad (31)$$

The time evolution of parameters vector is supposed to be random walk:

$$\vec{w}_i = \vec{w}_{i-1} + \vec{e}_{w,i} \quad (32),$$

where noise vector $\vec{e}_{w,i}$ is supposed to be Gaussian with zero mean and covariance W:

$$\mathrm{cov}[\vec{e}_{w,i},\vec{e}_{w,j}] = 0 \quad i \neq j, \quad \mathrm{cov}[\vec{e}_{w,i},\vec{e}_{w,j}] = W \quad i = j \quad (33).$$

The extended Kalman estimation filter could be for the studied case written in following form:

$$\begin{aligned} \vec{w}_i &= \vec{w}_{i-1} + H_i \bullet (\vec{z}_{i-1} - D_{i-1} \bullet \vec{w}_{i-1}) \\ S_i &= S_{i-1} + Q \\ H_i &= S_{i-1} \bullet D_{i-1}^T \bullet (D_{i-1} \bullet S_i \bullet D_{i-1}^T + W)^{-1} \end{aligned} \quad (34)$$

where a priory information $\vec{w}_1, S_1$ must be known (estimated) in advanced.

8. Examples of low dimensional identification

Let us define the LTI system with two repeated eigenvalues $\lambda_1 = \lambda_2 = 1$ as follows with matrix (23)

$$\begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \bullet \begin{bmatrix} x_{1,i-1} \\ x_{2,i-1} \end{bmatrix} + \begin{bmatrix} e_{1,i} \\ e_{2,i} \end{bmatrix}, \quad \begin{bmatrix} x_{1,0} \\ x_{2,0} \end{bmatrix} = \begin{bmatrix} 0.5 \\ -2 \end{bmatrix} \quad (35)$$

$$\begin{bmatrix} r_{1,1} & r_{1,2} \\ r_{2,1} & r_{2,2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad (36)$$

In this example the parameter vector $\vec{w} = (a_1, b_1, \alpha_1, a_2, b_2, \alpha_2)$ will be estimated by identification method. The equations (26) could be rewritten for studied example as follows (in this example m=2, n=4):

$$\begin{bmatrix} f_1(\vec{x}_i,\vec{w}_{i-1}) \\ f_2(\vec{x}_i,\vec{w}_{i-1}) \\ f_3(\vec{x}_i,\vec{w}_{i-1}) \\ f_4(\vec{x}_i,\vec{w}_{i-1}) \end{bmatrix} = \begin{bmatrix} \hat{a}_{1,i-1}x_{1,i} + \hat{b}_{1,i-1}x_{2,i} - \hat{\alpha}_{1,i-1} \bullet (\hat{a}_{1,i-1} \bullet x_{1,i-1} + \hat{b}_{1,i-1} \bullet x_{2,i-1}) \\ \hat{a}_{2,i-1}x_{1,i} + \hat{b}_{2,i-1}x_{2,i} - \hat{\alpha}_{2,i-1} \bullet (\hat{a}_{2,i-1} \bullet x_{1,i-1} + \hat{b}_{2,i-1} \bullet x_{2,i-1}) \\ (\hat{a}_{1,i-1} + \hat{a}_{2,i-1}) \bullet x_{1,i} + (\hat{b}_{1,i-1} + \hat{b}_{2,i-1}) \bullet x_{2,i} - x_{1,i} \\ (\hat{a}_{1,i-1} - \hat{a}_{2,i-1}) \bullet x_{1,i} + (\hat{b}_{1,i-1} - \hat{b}_{2,i-1}) \bullet x_{2,i} - x_{1,i} \end{bmatrix} \quad (37)$$

where $\vec{w}_{i-1} = (\hat{a}_{1,i-1}, \hat{b}_{1,i-1}, \hat{\alpha}_{1,i-1}, \hat{a}_{2,i-1}, \hat{b}_{2,i-1}, \hat{\alpha}_{2,i-1})$ is last estimate of parameters vector. In simulation mode the noise covariance was selected:

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (38)$$

In Fig. 1 the approximate and original evolution of state component $x_{1,i}$ is shown where the evolution of estimated vector parameters is shown in Fig. 2.

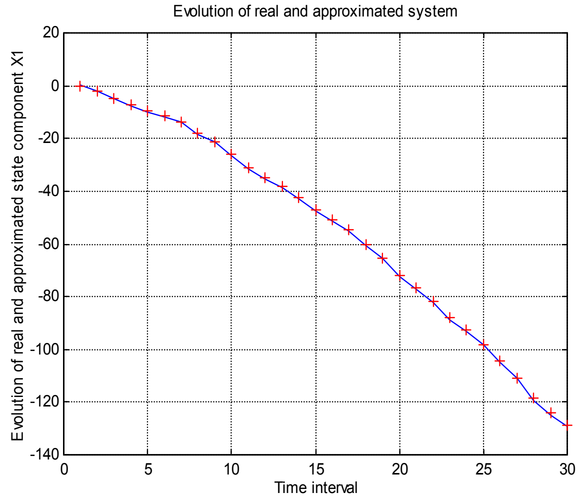

Fig.1 Approximate (+) and original evolution of state components $x_{1,i}$

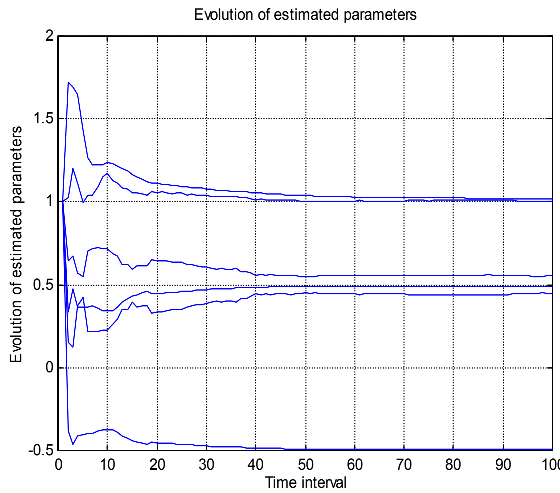

Fig. 2 Evolution of estimated parameters with result:

$\hat{a}_{1,100} = 0.4457$, $\hat{b}_{1,100} = 0.5540$,

$\hat{\cdot}_{1,100} = 1.0064$, $\hat{a}_{2,100} = 0.4767$, $\hat{b}_{2,100} = -0.4768$,

$\hat{\cdot}_{2,100} = 0.9687$

The same method was used for system with two complex conjugate eigenvalues $\cdot_1 = 0.5 + j \bullet 0.866$, $\cdot_2 = 0.5 - j \bullet 0.866$ as follows

$$\begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \bullet \begin{bmatrix} x_{1,i-1} \\ x_{2,i-1} \end{bmatrix} + \begin{bmatrix} e_{1,i} \\ e_{2,i} \end{bmatrix}, \quad \begin{bmatrix} x_{1,0} \\ x_{2,0} \end{bmatrix} = \begin{bmatrix} 0.5 \\ -2 \end{bmatrix} \qquad (39)$$

Fig. 3 shows the evolution of approximate and original state component $x_{1,i}$ and Fig. 4 shows the evolution of parameters vector.

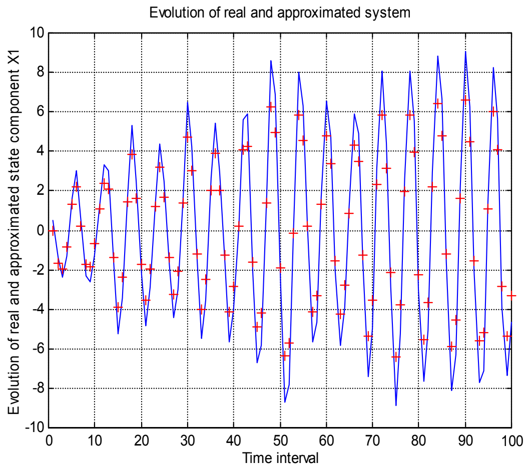

Fig.3 Approximate (+) and original evolution of state components $x_{1,i}$

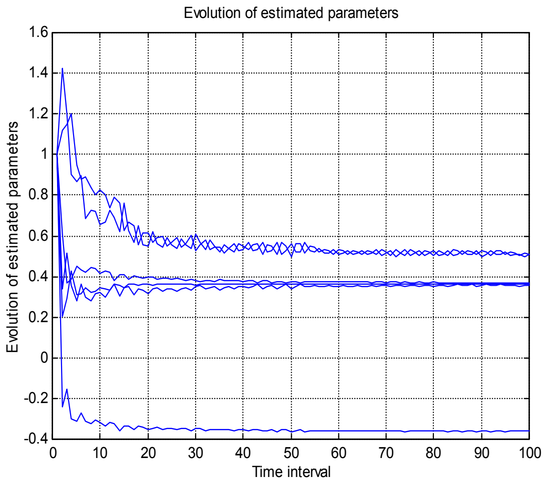

Fig.4 Evolution of estimated parameters with result: $\hat{a}_{1,100} = 0.3637$, $\hat{b}_{1,100} = 0.3626$,

$\hat{\cdot}_{1,100} = 0.5165$, $\hat{a}_{2,100} = 0.3619$,

$\hat{b}_{2,100} = -0.3588$, $\hat{\cdot}_{2,100} = 0.4923$

9. Conclusion

The presented results have shown the theory of LTI systems decomposition for distinct and repeated eigenvalues of transition matrix A of state-space model together with identification algorithm. This theory vindicates much known practice of mixtures of low dimensional dynamical models to approximate the higher order dynamical system. In paper the direct proof of LTI dynamical system decomposition with distinct and repeated eigenvalues of matrix A was presented as fundamental decomposition theorem. The decomposition theory was demonstrated on numerical example together with identification method.

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