# A Novel BMI Optimization Approach to Robust Controller Synthesis

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**Abstract:** In this paper an improved algorithm for real $\mu$ synthesis is presented. We propose a new BMI formulation which allows one to perform joint search for the controllers and the multipliers. This is of great difference from the conventional (D,G)-K iteration and its variations. The results can be extended to the other cases with different uncertainty descriptions.

**KEYWORDS**: Robust control, real $\mu$ synthesis, parametric uncertainty, generalized multiplier, bilinear matrix inequality (BMI), linear matrix inequality (LMI)

## I. INTRODUCTION

In the last two decades, $\mu$ analysis and synthesis have emerged to be a powerful tool for analyzing and designing robust controllers for systems subject to multiple sources of uncertainties. In the complex $\mu$ case, it is well known that $\mu$ controllers can be obtained by minimizing the $H_\infty$ norm of an appropriately scaled transfer matrix with respect to the multiplier and the controller [1-5] (a rigorous proof of certain important identities used in $\mu$ synthesis can be found in [6,7] ). Unfortunately, the synthesis problem is not jointly convex in the multiplier (or scaling) and the controller, though separately convex in each of these variables. Thus the currently existing synthesis is essentially based on iterating between the phase of computing the $\mu$ upper bound with the controller fixed, and the phase of $H_\infty$ optimization with the multiplier fixed.

In particular, in the phase of computing optimal multipliers, the scalings are available via solving a set of linear matrix inequalities at several grid frequencies, and curve fitting is performed to obtain a finite dimensional transfer function representation of them. In many cases, improvement of robustness during the $\mu$ synthesis iteration strongly depends on the quality of the curve fits for the scalings and this step has been seriously criticized as the weak link in $\mu$ synthesis. To alleviate this difficulty, Safonov and coworkers [4,8] proposed a multiplier approach (Km-synthesis or called M-K iteration) to compute suitable scalings. In their approach, the scaled $H_\infty$ norm minimization problem is transformed into an equivalent generalized positive real problem, in which the scalings are replaced with a linear parameterization of some fixed-order transfer functions satisfying certain properties. Thus no curve fitting of the scalings is required. A similar method employing rational functions as a basis was proposed in [9] at about the same time. These two methods fall into the category of the so called basis function method.

In the subsequent development for $\mu$ synthesis, Goh et al presented a bilinear matrix inequality (BMI) formulation for (mixed) $\mu$ synthesis [10]. This formulation allows the finite dimensional joint local and global optimization over an arbitrarily linear combination of a prescribed set of basis multipliers and the controller space. This result by this approach always improves on the conventional (D,G)-K iteration and M-K iteration. However, the iterative scheme for solving the generalized multiplier in one phase and the controller in another phase is retained. This motivates the present research of developing new computational algorithms for real $\mu$ synthesis on the basis of a new BMI formulation, which allows joint search of the generalized multiplier and the controller in a single phase.

The paper is organized as follows. Section II gives the problem statement and some preliminaries for future developments. Section III presents the main results. A new BMI formulation and the induced algorithm are given. Section IV illustrates this approach by a numerical example. Section V is the conclusions.

## II. PROBLEM FORMULATION AND PRELIMINARIES

### A. NOTATION

$RF$ := the set of real-rational, proper transfer functions.

$RH_\infty$:= the set of transfer matrices whose entries are proper, real-rational functions with no poles on the closed right half complex plane.

Let $D \in RF$ , then $D^{\sim}(s) = D^T(-s)$ .

$\Delta_r := \{block-diag(\Delta_1,...,\Delta_l) : \Delta_i = \delta_i I_{n_i}, \delta_i \in R, i = 1,\ldots,l\}$

$n_r := \sum_{i=1}^{l} n_i$

$\Gamma = \begin{pmatrix} \gamma I_{n_r} & 0 \\ 0 & I_p \end{pmatrix}$, where $p$ is the number of signals injecting into the controller

$S = \begin{pmatrix} I_{n_r} & -\sqrt{2} I_{n_r} \\ \sqrt{2} I_{n_r} & -I_{n_r} \end{pmatrix}$ ( Sector transform)

$\tilde{P}_\gamma = S * \Gamma P$ where $*$ means star product

$H(s) \leftrightarrow \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ means $H(s) = C(sI - A)^{-1} B + D$

Herm$[X] = \frac{1}{2}(X + X^T)$

$F_l(\bullet,\bullet)$ : lower linear fractional transformation

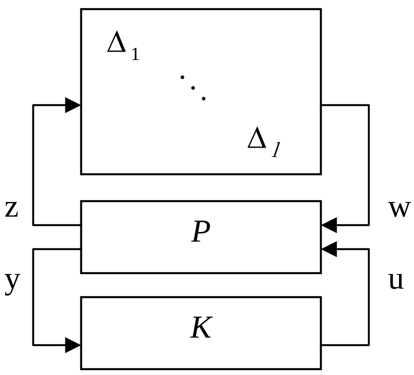


Fig. 1 $\Delta - P - K$ framework

### B. Problem formulation

Consider the robust controller synthesis framework in Fig. 1, where $P$ denotes the generalized plant described by

$$P \quad \begin{cases} \dot{x} = Ax + B_1 w + B_2 u \\ z = C_1 x + D_{11} w + D_{12} u \\ y = C_2 x + D_{21} w + D_{22} u \end{cases}$$

where $x \in R^n$ is the state vector, $w \in R^{n_r}$ denotes the exogenous signal, $u \in R^p$ is the control signal, $z \in R^{n_r}$ is the signal connected to the uncertainty, and $y \in R^p$ is the measurement output. $\Delta \in \Delta_r$ represents the structured parametric uncertainties of the control system, and $K$ is the dynamic output feedback controller of the following form to be designed.

$$K \quad \begin{cases} \dot{x}_K = A_K x_K + B_K y \\ u = C_K x_K + D_K y \end{cases}$$

The purpose of this paper is to find a dynamic output feedback controller such that the closed-loop system is uniformly robustly stable against as large parametric uncertainty $\Delta \in \Delta_r$ as possible. This is the general real $\mu$ controller synthesis problem. A number of sufficient conditions for the problem have appeared in the literature, notably [1,4,5,10], which were shown to be equivalent [6]. For the latter development of our results, the frequency domain condition presented in [10] is restated in the next theorem.

**Theorem 1** [10]: The nominal system $F_l(P,K)$ in Fig. 1 is uniformly robustly stable against the set of real parametric uncertainties $\Delta \in \Delta_r$ with sizes no greater than $\gamma$ if there exist a controller $K$ and generalized multiplier $W$ in $S_r(RF)$ satisfying

(i) $F_l(\tilde{P}_\gamma, K) \in RH_\infty$ ;

(ii) Herm$[W(j\omega)] > 0 \quad \forall \omega \in R \cup \infty$ ;

(iii) Herm$[F_l(\tilde{P}_\gamma, K) W(j\omega)] > 0 \quad \forall \omega \in R \cup \infty$ .

It is interesting to note that the transfer function $F_l(\tilde{P}_\gamma, K)W$ of condition (iii) in Theorem 1 can be interpreted as the map from $\bar{w}$ to $\bar{z}$ as shown below.

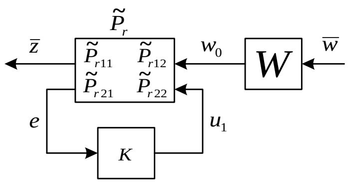


Fig. 2

With the linear fractional transformation technique, we can redraw Fig. 2 as Fig. 3. The

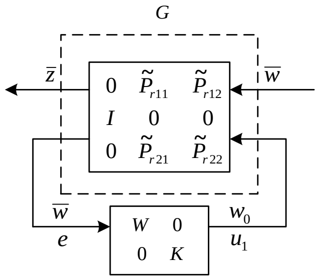


Fig. 3

equivalence is formally established in the following lemma, which is of paramount importance to our results in that it will induce a brand-new development for the general real $\mu$ controller synthesis problem.

**Lemma 1**: $F_l(\tilde{P}_\gamma, K)W = F_l(G, diag(W,K))$, where

$$G = \begin{bmatrix} 0 & \tilde{P}_{r11} & \tilde{P}_{r12} \\ I & 0 & 0 \\ 0 & \tilde{P}_{r21} & \tilde{P}_{22} \end{bmatrix}.$$

Furthermore, if $K = \frac{1}{s+a}K_1$ for some $a \in R$, then $F_l(\tilde{P}_\gamma, K)W = F_l(G_1, diag(W, K_1))$, where

$$G_1 = diag(I, I, \frac{1}{s+a})G. \qquad (1)$$

Proof: Draw the flow chart for the transfer function $F_l(\tilde{P}_\gamma, K)W$ and pull out $W$ and $K$.

## III. Main Results

In this section, we will reinterpret the robust stability condition of Theorem 1 by the result of Lemma 1. The impact of the new formulation lies on that it allows one to simultaneously compute the controller and the generalized multiplier in a single phase. This features a quite different synthesis framework from the conventional *D-K* (or (*D*,*G*)-*K*) iteration and its variations, which usually involves two phases between alternatively computing $H_\infty$ optimal controller and computing $\mu$ upper bound.

### A. Robust Stability Condition in New BMI Formulation

The robust stability condition of Theorem 1 is reinterpreted in new formulation as follows.

**Theorem 2:** The nominal system $F_l(P,K)$ in Fig. 1 is uniformly robustly stable against the set of real parametric uncertainties $\Delta \in \Delta_r$ with sizes no greater than $\gamma$ if there exist a transfer function $K_1$ and transfer matrix $W$ in $S_r(RF)$ such that

(i) $F_l(\hat{P}_\gamma, K_1) \in RH_\infty$;

(ii) Herm$[W(j\omega)] > 0 \quad \forall \omega \in R \cup \infty$;

(iii) Herm$[F_l(G_1, diag(W, K_1))(j\omega)] > 0 \quad \forall \omega \in R \cup \infty$.

where $\hat{P}_\gamma = diag(I, \frac{1}{s+a})\tilde{P}_\gamma$, and $G_1$ is described in (1).

Proof: By Theorem 1 and Lemma 1.

Next, we make use of the following generalized positive real lemma to yield BMI representation for the frequency domain condition of Theorem 2.

### B. Bilinear Matrix Inequality Representation for the Frequency Domain Condition

Lemma 2 [11]: Consider a plant $X(s) = C(sI-A)^{-1}B + D$, with none of the eigenvalue of $A$ lies on the imaginary axis of the complex plane. There exists $\varepsilon > 0$, $\kappa > 0$ such that

$$\kappa I \geq Herm\{X(j\omega)\} \geq \varepsilon I \quad \forall \omega \in R \cup \infty,$$

if there exist $P = P^T$ such that

$$\text{Herm}\left\{ \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right\} > 0.$$

Define the following notation:

$$P \leftrightarrow \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$$

Given $\gamma > 0$, then

$$\tilde{P}_r = S * \Gamma P = \left[\begin{array}{c|c} \tilde{P}_{r11} & \tilde{P}_{r12} \\ \hline \tilde{P}_{r21} & \tilde{P}_{r22} \end{array}\right] \leftrightarrow \left[\begin{array}{c|cc} \tilde{A} & \tilde{B}_1 & \tilde{B}_2 \\ \hline \tilde{C}_1 & \tilde{D}_{11} & \tilde{D}_{12} \\ \tilde{C}_2 & \tilde{D}_{21} & \tilde{D}_{22} \end{array}\right].$$

Assume $K = K_1 \times \frac{1}{s+a}$ for some $a \in R$, then

$$\hat{P}_r := diag\left(I, \frac{1}{s+a} I\right)\tilde{P}_r \leftrightarrow \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & \hat{D}_{11} & \hat{D}_{12} \\ \hat{C}_2 & \hat{D}_{21} & \hat{D}_{22} \end{array}\right]$$

(It is easy to verify that $\hat{D}_{22} = 0$).

$$G_1 \leftrightarrow \left[\begin{array}{c|c|c} \bar{A} & \bar{B}_1 & \bar{B}_2 \\ \hline \bar{C}_1 & \bar{D}_{11} & \bar{D}_{12} \\ \hline \bar{C}_2 & \bar{D}_{21} & \bar{D}_{22} \end{array}\right] \text{( see Lemma 1).}$$

These lead to the following BMI formulation (different from [10]) for the general real $\mu$ controller synthesis problem.

Problem: Given generalized plant $P$. Maximize the value $\gamma$ subject to the following BMIs (1)-(4), i.e., maximize $\gamma$ subject to the existence of the matrix variables $P_M = P_M^T$, $P_W = P_W^T$, $P = P^T$, ($A_{K_1}$, $B_{K_1}$, $C_{K_1}$, $D_{K_1}$), ($A_{W_i}$, $B_{W_i}$, $C_{W_i}$, $D_{W_i}$), $i = 1,\ldots,l$, and a scalar $a$ satisfying the following BMIs

$$P_M > 0 \quad (1)$$

$$\text{Herm}\{P_M(R_A + U_A Q_{K_1} V_A)\} < 0 \quad (2)$$

$$\text{Herm}\left\{\begin{bmatrix} P_W & 0 \\ 0 & -I \end{bmatrix} Q_W\right\} < 0 \quad (3)$$

$$\text{Herm}\left\{\begin{bmatrix} P & 0 \\ 0 & -I \end{bmatrix}(R_T + U_T Q_T V_T)\right\} < 0 \quad (4)$$

where

$$R_A = \begin{bmatrix} 0_{q_{k_1}} & 0 \\ 0 & \hat{A} \end{bmatrix};\ U_A = \begin{bmatrix} I_{q_{k_1}} & 0 \\ 0 & \hat{B}_2 \end{bmatrix};\ V_A = \begin{bmatrix} I_{q_{k_1}} & 0 \\ 0 & \hat{C}_2 \end{bmatrix},$$

where $q_{k_1}$ is the order of the controller $K_1$.

$$R_T = \begin{bmatrix} 0_q & 0 & 0 \\ 0 & \bar{A} & \bar{B}_1 \\ 0 & \bar{C}_1 & \bar{D}_{11} \end{bmatrix};\ U_T = \begin{bmatrix} I_q & 0 \\ 0 & \bar{B}_2 \\ 0 & \bar{D}_{12} \end{bmatrix};$$

$$V_T = \begin{bmatrix} I_q & 0 & 0 \\ 0 & \bar{C}_2 & \bar{D}_{21} \end{bmatrix};$$

where $q =$ $q_{k_1}$ plus the order of the generalized multiplier $W$.

$$Q_{K_1} := \begin{bmatrix} A_{K_1} & B_{K_1} \\ C_{K_1} & D_{K_1} \end{bmatrix} \leftrightarrow K_1.$$

$$Q_W := \begin{bmatrix} A_W & B_W \\ C_W & D_W \end{bmatrix} \text{ with}$$

$$\begin{bmatrix} A_W & B_W \\ C_W & D_W \end{bmatrix} := \left[\begin{array}{ccc|ccc} A_{W_1} & 0 & 0 & B_{W_1} & 0 & 0 \\ 0 & \ddots & 0 & 0 & \ddots & 0 \\ 0 & 0 & A_{W_l} & 0 & 0 & B_{W_l} \\ \hline C_{W_1} & 0 & 0 & D_{W_1} & 0 & 0 \\ 0 & \ddots & 0 & 0 & \ddots & 0 \\ 0 & 0 & C_{W_l} & 0 & 0 & D_{W_l} \end{array}\right].$$

$$Q_T := \left[\begin{array}{cc|cc} A_W & 0 & B_W & 0 \\ 0 & A_{K_1} & 0 & B_{K_1} \\ \hline C_W & 0 & D_W & 0 \\ 0 & C_{K_1} & 0 & D_{K_1} \end{array}\right].$$

## C. New Real Mu Synthesis Algorithm

Step 1. Compute the generalized plant $P(s)$.

Step 2. Get a multiplier $D$ and a controller $K_{dkit}$ from the i-th *D-K* iteration by the "dkit" command of $\mu$-Analysis and Synthesis Toolbox [2].

Step 3. Compute $\gamma = 1/\|DMD^{-1}\|_\infty$ and set the initial generalized multiplier $W_{dkit} = DD^{\sim}$.

Assume $K = K_{dkit} = \frac{1}{s+a} K_1$ for some $a \in R$

(If this is not the case, we could modify the algorithm by inserting a strictly proper internal model into the closed loop before forming the $\Delta - P - K$ framework in order to fulfill the requirement here).

Step 4. Fix the controller $K_1$ and the generalized multiplier $W = W_{dkit}$, solve LMI (1)-(4) for the auxiliary variables $(P_M, P_W, P)$:

Step 4.1. Increase $\gamma$. Compute $\tilde{P}_r$, $R_A$, $U_A$, $V_A$, $R_T$, $U_T$, $V_T$, $Q_{K_1}$, $Q_W$, and $Q_T$.

Step 4.2. Solve the LMIs (1)-(4) for the auxiliary variables $(P_M, P_W, P)$.

Step 4.3. Repeat Step 4.1 and Step 4.2 till the LMIs are near infeasible.

Step 5. Fix the auxiliary variables $(P_M, P_W, P)$, solve LMIs (2)-(4) for new controller and generalized multiplier, i.e., ($A_{K_1}$, $B_{K_1}$, $C_{K_1}, D_{K_1}$), ($A_{W_i}$, $B_{W_i}$, $C_{W_i}$, $D_{W_i}$), $i = 1,\ldots,l$:

Step 5.1. Increase $\gamma$. Compute $\tilde{P}_r$, $R_A$, $U_A$, $V_A$, $R_T$, $U_T$, and $V_T$.

Step 5.2. Solve (2)-(4) for new controller and generalized multiplier, i.e., ($A_{K_1}$, $B_{K_1}$, $C_{K_1}$, $D_{K_1}$), and ($A_{W_i}$, $B_{W_i}$, $C_{W_i}$, $D_{W_i}$), $i=1,\ldots,l$.

Step 5.3. Repeat Step 5.1 and Step 5.2 till the LMIs are near infeasible.

Step 6. Iteratively perform Step 4 and Step 5 till there is no significant increase in $\gamma$.

Step 7. The resulting controller is given by $K = \frac{1}{s+a}K_1$.

## IV. NUMERICAL EXAMPLE

Example: Consider the generalized plant

$$P(s) \leftrightarrow \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right].$$

$$= \left[\begin{array}{cc|ccccc} -4 & 3 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ -1 & -3 & -1 & 0 & -1 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ -1 & -3 & -1 & 0 & -1 & 0 & 0 \end{array}\right]$$

subject to parametric uncertainty $\Delta = diag(\delta_1, \delta_2, \delta_3, \delta_4)$.

Three robust controller design methods are carried out to yield the following results:

A. Matlab command "hinflmi"

$$\gamma_{\mathbf{inf}} = \frac{1}{4.6302} = 0.21597$$

$$K_{\mathbf{inf}} = \frac{2.036s + 9.3639}{s + 3.8592}$$

B. *D-K* iteration ("dkit" command of [2])

$$\gamma_{dkit} = \frac{1}{4.23} = 0.23641$$

$$K_{dkit} = \frac{469.8s^5 + 2.3239\times10^5 s^4 + 2.7709\times10^7 s^3}{s^6 + 697.18s^5 + 1.6232\times10^5 s^4 + 1.2901\times10^7 s^3} \frac{+1.6607\times10^8 s^2 + 2.1811\times10^8 s + 7.9265\times10^7}{+6.5695\times10^7 s^2 + 8.2862\times10^7 s + 2.9655\times10^7}$$

C. Our approach

By our method, we obtain that

$$\gamma_{our} = \frac{1}{4.0427} = 0.24736$$

$$K_{our} = \frac{343.14s^5 + 5.4738\times10^5 s^4 + 1.1168\times10^8 s^3}{s^6 + 1347.2s^5 + 4.8467\times10^5 s^4 + 5.463\times10^7 s^3} \frac{+1.2838\times10^9 s^2 + 4.209\times10^9 s + 3.6144\times10^9}{+5.4827\times10^8 s^2 + 1.6221\times10^9 s + 1.3193\times10^9}$$

Notice that the stability margin guaranteed by our method is better than that by $H_\infty$ control and that by *D-K* iteration using currently available software [2].

## V. CONCLUSIONS

We have presented a new real $\mu$ synthesis method which can be viewed as an alternative/supplement to the conventional D-K (or (D,G)-K) iteration or the other $\mu$ synthesis methods. The advantages of this new algorithm are as follows: (1) performs simultaneous search for the controllers and multipliers. Particularly, the search for the poles and the zeros of the multipliers can be conducted simultaneously (different from the existing approaches, e.g., [8,9,10], where the poles were fixed); (2) no need of curve fitting; (3) no need of spectral factorization; (4) the orders of the controllers and multipliers are fixed during the design. Moreover, it always improves on the results obtained by the other $\mu$ synthesis schemes. Finally, the approach can easily be extended to the other cases with different uncertainty descriptions, which will appear in the coming papers.

## VI. ACKNOWLEDGEMENT

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