Trellis properties on the tensor product of two lattices*

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Abstract: - Trellis diagrams of lattices and the Viterbi algorithm can be used for decoding. In the paper, we investigate some relations among the trellis diagrams of the lattices L_1, L_2 and $L_1 \otimes L_2$.

Key-Words: - Lattice, Trellis diagrams, Tensor product.

1 Introduction and Preliminaries

It is well known that trellis diagrams can been employed for maximum-likelihood decoding by using the Viterbi algorithm ([2]). The simpler the trellis diagram is, the more efficient decoding is. There are several methods constructing the minimal trellis diagrams for linear codes ([10]). G. D. Forney defined the trellis diagram of lattice in [3] and [4], which can be used for decoding of the code based on lattices. The complexity of the trellis diagram of a lattice is generally measured by the numbers of states, edges and labels at every level. Up to now, no efficient methods are found to construct trellis diagrams with low complexity for a lattice. However, V. Tarokh extensively studied the trellis complexity of lattices in [7], [8] and [9], whose results are profound. After having read Tarokh's Ph.d thesis, G. D. Forney thought Tensor product would be an important tool in studying the trellis complexity of lattices. For example, we can construct lattices with arbitrary large code gain by tensor product; Some interesting lattices including the Barnes-Wall lattices can be constructed using tensor product. In the paper, we study the trellis relations among lattices L_1, L_2 and their tensor product $L_1 \otimes L_2$.

Denote by R and Z the sets of real numbers and integers, respectively. Let R^n and $R^{m\times n}$ be the set of real n-dimensional column vectors and that of $m \times n$ matrices with elements in R, respectively. All vectors are assumed to be column vectors. For integers $x_1, x_2, \cdots, x_i$, we use $(x_1, x_2, \cdots, x_i)$ for the greatest common divisor and $[x_1, x_2, \cdots, x_i]$ for the least common multiple of $x_1, x_2, \cdots, x_i$. A lattice is a discrete and additive subgroup in R^n. Concretely, For any linearly independent vectors $b_1, b_2, \cdots, b_m \in R^n$, $m \le n$, the set $L = \{\sum_{i=1}^{m} k_i b_i \mid k_i \in Z, 1 \le i \le m\}$ is called a lattice. $(b_1, b_2, \cdots, b_m)$ is called a basis of the lattice L, m the dimension of L. We also use the notations $L(b_1, b_2, \cdots, b_m)$ or $L(B)$ for the lattice L, where the matrix $B = (b_1, b_2, \cdots, b_m)$ is

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called a basis matrix of L. If $m = n$, L is a full rank lattice. In the paper, we always assume lattices are of full rank. For vectors $u = (u_1, u_2, \cdots, u_n)^t$, $v = (v_1, v_2, \cdots, v_n)^t \in R^n$, their inner product is defined as $\langle u, v\rangle = \sum_{i=1}^{n} u_i v_i$. u and v are orthogonal if $\langle u, v\rangle = 0$. Denote by $span(b_1, b_2, \cdots, b_m)$ the vector space spanned by the vectors $b_1, b_2, \cdots, b_m$.

For any subspace $W \subseteq R^n$, there exists a unique orthogonal complement $W^\perp$, i.e., $R^n = W \oplus W^\perp$. Let P_W be the projection of R^n in W. This is, for any $x \in R^n$, $P_W(x) = w$, where $x = w + u$, $w \in W$, $u \in W^\perp$. Clearly P_W is linear. The lattice L has a finite trellis if it has n pairwise orthogonal elements. Let $w_1, \cdots, w_n$ be primitive and pairwise orthogonal vectors in L, i.e., $\langle w_i, w_j\rangle = 0$ for $1 \le i < j \le n$, and $L \cap span(w_i) = L(w_i)$ for $1 \le i \le n$, where $span(w_i)$ is the subspace spanned by w_i and $L(w_i)$ is the sublattice spanned by w_i. Let $W_i = span(w_i)$ and $V_i = W_1 \oplus W_2 \oplus \cdots W_i$, $1 \le i \le n$. For any subspace V in R^n, let $P_V : \Lambda \longmapsto V$ be the projection of Λ in V and $L_V = L \cap V$. The state space at time i is $\Sigma_i(L) = P_{V_i}(L)/L_{V_i}$, and the label space at time i is $G_i(L) = P_{W_i}(L)/L_{W_i}$. For $x \in L$, define $\sigma(x) = (\sigma_0(x), \sigma_1(x), \cdots, \sigma_n(x))$ and $g(x) = (g_1(x), g_2(x), \cdots, g_n(x))$, where $\sigma_i(x) = P_{V_i}(x) + L_{V_i} \in \Sigma_i(L)$ and $g_i(x) = P_{W_i}(x) + L_{W_i} \in G_i(L)$, $1 \le i \le n$. A trellis T for L under the coordinate system $\{W_i\}_{i=1}^n$ is an edge-labeled directed graph, whose i-th level nodes are the elements in $\Sigma_i(L)$, and whose edges from the i-th level nodes to the $(i+1)$-th level nodes are $\{(\sigma_i(x), g_{i+1}(x), \sigma_{i+1}(x)) | x \in \Lambda\}$. If we denote by W the coordinate system $\{W_i\}_{i=1}^n$, then let $s_i(L, W) = |\Sigma_i(L)|$ and $g_i(L, W) = |G_i(L)|$, $e_i(L, W) = |E_i(L)|$, $0 \le i \le n-1$, where $E_i(L) = \{(\sigma_i(x), g_{i+1}(x), \sigma_{i+1}(x)) | x \in \Lambda\}$. Denote by $N(L, W)$ the number of distinct paths from the initial state to the end state of T. When there is no ambiguity, we simply denote the above notations by s_i, g_i, e_i and $N(L)$.

Given two vectors $v = (v_1, v_2, \cdots, v_m)^t \in R^m, u = (u_1, u_2, \cdots, u_n)^t \in R^n$, define $v \otimes u = (v_1u_1, v_1u_2, \cdots, v_1u_n, v_2u_1, v_2u_2, \cdots, v_2u_n, \cdots, v_mu_1, v_mu_2, \cdots, v_mu_n)^t \in R^{m+n}$. Call $v \otimes u$ the tensor product of v and u. For any real $a \in R$, define $a \cdot v = (av_1, av_2, \cdots, av_m)^t$. Sometimes we write $v \cdot a$ for $a \cdot v$. It is easy to verify that $(v \cdot a) \otimes u = v \otimes (a \cdot u)$. Similarly, we define the tensor product of two matrices to be $B \otimes B' = (b_1 \otimes b'_1, b_1 \otimes b'_2, \cdots, b_1 \otimes b'_{n'}, \cdots, b_n \otimes b'_1, b_n \otimes b'_2, \cdots, b_n \otimes b'_{n'}) \in R^{mm' \times nn'}$, where $B = (b_1, b_2, \cdots, b_n) \in R^{m \times n}$, $B' = (b'_1, b'_2, \cdots, b'_{n'}) \in R^{m' \times n'}$. For $u, v \in R^m$, $x, y \in R^n$, by the definition of the tensor product of two vectors, we have $(u+v) \otimes x = u \otimes x + v \otimes x, u \otimes (x+y) = u \otimes x + u \otimes y$, and $\langle u \otimes x, v \otimes y\rangle = \langle u, v\rangle \cdot \langle x, y\rangle$. For any two lattices L_1 and L_2, define the tensor product $L_1 \otimes L_2 = \{\sum_{i=1}^{s} v_i \otimes u_i | v_i \in L_1, u_i \in L_2, 1 \le i \le s$, s is an arbitrary positive integer$\}$. Clearly $L_1 \otimes L_2$ is a lattice.

2 Trellis relations between the lattices L_1, L_2 and $L_1 \otimes L_2$.

In this section, we investigate the relations of the path numbers, degree numbers and state numbers in trellis diagrams of the lattices L_1, L_2 and $L_1 \otimes L_2$.

Lemma 2.1 *[6] Let $L \in \mathcal{L}_n$ have a finite trellis diagram under the coordinate system $\{W_i\}_{i=1}^n$, where $L \cap W_i = L(w_i)$, $1 \le i \le n$. Then there exists a basis $(b_1, b_2, \cdots, b_n)$ of L such that $(w_1, w_2, \cdots, w_n) = (b_1, b_2, \cdots, b_n)P$, where $P = (p_{ij})_{n \times n}$ is an upper triangular integer matrix. Furthermore, p_{ii} is the indegree of any vertex in the i-th level of the trellis diagram of L under the coordinate system $\{W_i\}_{i=1}^n$.*

It is easy to verify that the former part of the above Lemma is equivalent to lemma 2 of [1], and the later part of the lemma can be found in [6]. So we omit the proof of the above lemma.

Lemma 2.2 *Let $L_1 \subseteq R^n$ be a lattice with basis $B = (b_1, b_2, \cdots, b_n)$ and $L_2 \subseteq R^{n'}$ a lattice with basis $B' = (b'_1, b'_2, \cdots, b'_{n'})$. Then*

(1). $L_1 \otimes L_2$ is a lattice with basis $(b_1 \otimes b'_1, b_1 \otimes b'_2, \cdots, b_1 \otimes b'_{n'}, \cdots, b_n \otimes b'_1, b_n \otimes b'_2, \cdots, b_n \otimes b'_{n'})$, i.e., $L_1 \otimes L_2 = L(B \otimes B')$

(2). If L_1, $L_2 \in \pounds_n$, then $L_1 \otimes L_2 \in \pounds_n$, i.e., if L_1 and L_2 have finite trellis diagrams, then so does $L_1 \otimes L_2$.

Proof. (1) can be verified directly. (2) is the lemma 5.2 of [7]. ■

Lemma 2.3 *Let L_1 be an n-dimensional lattice and L_2 an n'-dimensional lattice. Then $\det(L_1 \otimes L_2) = \det(L_1)^{n'} \cdot \det(L_2)^n$.*

Proof. Let $L_1 = L(b_1, b_2, \cdots, b_n)$ and $L_2 = L(b'_1, b'_2, \cdots, b'_{n'})$. Since every lattice has a basis with Hermite Normal Form (low triangular matrix), we assume that the matrices $B = (b_1, b_2, \cdots, b_n)$ and $B' = (b'_1, b'_2, \cdots, b'_{n'})$ are matrices with Hermite Normal Forms. Let $B = (b_{ij})_{n \times n}$, where $b_{ij} = 0$ for $1 \leq i < j \leq n$. Clearly,

$$B \otimes B' = \begin{pmatrix} b_{11}B' & b_{12}B' & \cdots & b_{1n}B' \\ b_{21}B' & b_{22}B' & \cdots & b_{2n}B' \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1}B' & b_{n2}B' & \cdots & b_{nn}B' \end{pmatrix}.$$

By the definition of the tensor product, $B \otimes B'$ is a low triangular matrix. Hence $\det(B \otimes B') = (\det(B))^{n'} \cdot (\det(B'))^n$. ■

Lemma 5.4 of [7] is a special case of the above lemma. The following two Lemmas can be verified directly.

Lemma 2.4 *Let A, A' and Q be matrices such that $A = A'Q$, and B, B' and Q' matrices such that $B = B'Q'$. Then $A \otimes B = (A' \otimes B')(Q \otimes Q')$.*

Lemma 2.5 *Let $S, U \subseteq R^n$ and $T \subseteq R^m$ be subspaces, $L_1 \subseteq R^n$ and $L_2 \subseteq R^m$ be lattices. Then*

(1). $P_{S \otimes T}(L_1 \otimes L_2) = P_S(L_1) \otimes P_T(L_2)$, where P_S, P_T and $P_{S \otimes T}$ are the projections in S, T and $S \otimes T$, respectively.

(2). If S and U are orthogonal subspaces of R^n, then $P_{S \oplus U}(L_1) \subseteq P_S(L_1) \oplus P_U(L_1)$, where $P_{S \oplus U}$ is the projection in $S \oplus U$.

Proposition 2.6 *Let T_1 and T_2 be finite trellis diagrams of n-dimensional lattice L_1 and m-dimensional lattice L_2 under the coordinate systems $\{W_i\}_{i=0}^{n-1}$ and $\{U_i\}_{i=0}^{m-1}$, respectively, where $L_1 \cap W_i = L(w_i)$, $0 \leq i \leq n-1$, $L_2 \cap U_j = L(u_j)$, $0 \leq j \leq m-1$. Let $(a_0, a_2, \cdots, a_{n-1})$ be a basis of L_1 and $(b_0, b_1, \cdots, b_{m-1})$ a basis of L_2 such that $(w_0, w_1, \cdots, w_{n-1}) = (a_0, a_1, \cdots, a_{n-1})P$ and $(u_0, u_1, \cdots, u_{m-1}) = (b_0, b_1, \cdots, b_{m-1})Q$, where $P = (p_{ij})_{0 \leq i,j \leq n-1}$ and $Q = (q_{ij})_{0 \leq i,j \leq m-1}$ are upper triangular integer matrices. Let T be the finite trellis diagram of $L_1 \otimes L_2$ under the coordinate system $\{H_i\}_{i=0}^{mn-1}$, where $H_i = W_\alpha \otimes U_\beta$, $i = \alpha m + \beta$, $0 \leq \beta < m$. Then,*

(1). $N(L_1 \otimes L_2, T) = N(L_1, T_1)^m \cdot N(L_2, T_2)^n$, where $N(L_1 \otimes L_2, T)$, $N(L_1, T_1)$ and $N(L_2, T_2)$ mean the numbers of distinct paths in the trellis diagrams T, T_1 and T_2, respectively.

(2). $d_i^- = d_\alpha'^- \cdot d_\beta''^-$, where $0 \leq i < mn$, $i = \alpha m + \beta$, $0 \leq \beta < m$, and d_i^-, $d_\alpha'^-$ and $d_\beta''^-$ denote the in-degrees of vertices at the i-th, α-th and β-th levels of T, T_1 and T_2, respectively.

(3). $g_i = g'_\alpha \cdot g''_\beta$, where $0 \leq i < mn$, $i = \alpha m + \beta$, $0 \leq \beta < m$, and g_i, g'_α and g''_β denote the orders of label groups at the i-th, α-th and β-th levels of T, T_1 and T_2, respectively.

Proof. Clearly, $(L_1 \otimes L_2) \cap (W_\alpha \otimes U_\beta) = (L_1 \cap W_\alpha) \otimes (L_2 \cap U_\beta) = L(w_\alpha \otimes u_\beta)$. Since $(w_0, w_1, \cdots, w_{n-1}) = (a_0, a_1, \cdots, a_{n-1})P$ and $(u_0, u_1, \cdots, u_{m-1}) = (b_0, b_1, \cdots, b_{m-1})Q$, $(w_0 \otimes u_0, \cdots, w_{n-1} \otimes u_{m-1}) = (a_0 \otimes b_0, \cdots, a_{n-1} \otimes$

$b_{m-1})(P \otimes Q)$. Hence,

$$\begin{aligned} N(L_1 \otimes L_2, T) &= \frac{\det(w_0 \otimes u_0, \cdots, w_{n-1} \otimes u_{m-1})}{\det(a_0 \otimes b_0, \cdots, a_{n-1} \otimes b_{m-1})} \\ &= \det(P \otimes Q) \\ &= N(L_1, T_1)^m \cdot N(L_2, T_2)^n. \end{aligned}$$

Because the i-th element in the main diagonal of $P \otimes Q$ is $p_{\alpha\alpha} q_{\beta\beta}$, where $0 \leq i < mn$, $i = \alpha m + \beta$, $0 \leq \beta < m$, so $d_i^- = d_\alpha'^- \cdot d_\beta''^-$ by the Lemma 2.1. Obliviously, $(a_0 \otimes b_0, \cdots, a_{n-1} \otimes b_{m-1}) = (w_0 \otimes u_0, \cdots, w_{n-1} \otimes u_{m-1}) \cdot (P^{-1} \otimes Q^{-1})$. For any $0 \leq i < mn$, $i = \alpha m + \beta$, $0 \leq \beta < m$, Since the i-th row of $(P^{-1} \otimes Q^{-1})$ is the tensor product of the α-th row of P^{-1} and the β-th row of Q^{-1}, it is easy to verify that the least common multiple of denominators of the elements in the i-th row of $(P^{-1} \otimes Q^{-1})$ is the product of the least common multiples of denominators of the elements in the α-th row of P^{-1} and in the β-th row of Q^{-1}. Hence, (3) follows from $(*)$. ∎

Remark 2.7 *It is also not difficult to prove $d_i^+ = d_\alpha'^+ \cdot d_\beta''^+$, where $0 \leq i < mn$, $i = \alpha m + \beta$, $0 \leq \beta < m$, and d_i^+, $d_\alpha'^+$ and $d_\beta''^+$ denote the out-degrees of vertices at the i-th, α-th and β-th levels of T, T_1 and T_2, respectively.*

The result 3 of the above Proposition is in fact Lemma 5.3 of [7], here we deal with it in different view. The following Theorem gives the relation among the numbers of the states in the finite trellis diagrams of the lattices L_1, L_2 and $L_1 \otimes L_2$.

Theorem 2.8 *Let T_1 and T_2 be finite trellis diagrams of n-dimensional lattice L_1 and m-dimensional lattice L_2 under the coordinate systems $\{W_i\}_{i=0}^{n-1}$ and $\{U_j\}_{j=0}^{m-1}$, respectively, where $L \cap W_i = L(w_i)$, $0 \leq i \leq n-1$, $L \cap U_j = L(u_j)$, $0 \leq j \leq m-1$. Let $(a_0, a_1, \cdots, a_{n-1})$ be a basis of L_1 and $(b_0, b_1, \cdots, b_{m-1})$ a basis of L_2 such that $(w_0, w_1, \cdots, w_{n-1}) = (a_0, a_1, \cdots, a_{n-1})P$ and $(u_0, u_1, \cdots, u_{m-1}) = (b_0, b_1, \cdots, b_{m-1})Q$, where $P = (p_{ij})_{0 \leq i,j \leq n-1}$ and $Q = (q_{ij})_{0 \leq i,j \leq m-1}$ are upper triangular integer matrices. Let T be the finite trellis diagram of $L_1 \otimes L_2$ under the coordinate system $\{H_i\}_{i=0}^{mn-1}$, where $H_i = W_\alpha \otimes U_\beta$, $i = \alpha m + \beta$, $0 \leq \beta < m$. Denote by s'_α, s''_β and s_i the numbers of states at the α-th, β-th and i-th level of T_1, T_2 and T, respectively. Then,*

(1). $s_i = (s'_\alpha)^m$ if $i = \alpha m + (m-1)$.

(2). $s_i = (s'_0)^{\beta+1}(s''_\beta)$ if $i = 0 \cdot m + \beta$, i.e., $i < m$.

(3). $s_i \leq (s'_{\alpha-1})^m \cdot (\frac{g'_\alpha}{d_\alpha'^-})^{\beta+1} \cdot (s''_\beta)$, where $\alpha \geq 1$, g'_α is the order of the label group at the α-th level of T_1, and $d_\alpha'^-$ is the in-degree of any vertex at the α-th level of T_1.

Proof. For any $0 \leq i < mn$, $i = \alpha m + \beta$, $0 \leq \beta < m$, let

$V_i = span(w_0 \otimes u_0, \cdots, w_0 \otimes u_{m-1}, w_{\alpha-1} \otimes u_0, \cdots, w_{\alpha-1} \otimes u_{m-1}, w_\alpha \otimes u_0, \cdots w_\alpha \otimes u_\beta)$,
$F_\alpha = span(w_0, w_1, \cdots w_\alpha)$,
$G_\beta = span(u_0, u_1, \cdots u_\beta)$.

If $i = \alpha m + (m-1)$, then $V_i = span(w_0, \cdots, w_\alpha) \otimes span(u_0, \cdots, u_{m-1})$ $= span(a_0, \cdots, a_\alpha) \otimes span(L_2)$. So $V_i \cap (L_1 \otimes L_2) = L(a_0, \cdots, a_\alpha) \otimes L_2$ and $P_{V_i}(L_1 \otimes L_2) = P_{span(w_0, \cdots, w_\alpha)}(L_1) \otimes P_{span(L_2)}(L_2)$ $= P_{F_\alpha}(L_1) \otimes L_2$. Therefore, $P_{V_i}(L_1 \otimes L_2)/(V_i \cap (L_1 \otimes L_2)) = (P_{F_\alpha}(L_1) \otimes L_2)/(L(a_0, \cdots, a_\alpha) \otimes L_2)$. Consequently,

$$\begin{aligned} s_i &= \frac{\det(L(a_0, \cdots, a_\alpha) \otimes L_2)}{\det(P_{F_\alpha}(L_1) \otimes L_2)} \\ &= \left(\frac{\det L(a_0, \cdots, a_\alpha)}{\det P_{F_\alpha}(L_1)}\right)^m \\ &= (s'_\alpha)^m, \end{aligned}$$

and (1) holds.

If $i = 0 \cdot m + \beta$, i.e., $i < m$, then

$$\begin{aligned} V_i &= span(w_0 \otimes u_0, \cdots, w_0 \otimes u_\beta) \\ &= span(a_0 \otimes b_0, \cdots, a_0 \otimes b_\beta). \end{aligned}$$

Thus, $V_i \cap (L_1 \otimes L_2) = L(a_0) \otimes L(b_0, \cdots, b_\beta)$ and $P_{V_i}(L_1 \otimes L_2) = P_{span(w_0)}(L_1) \otimes P_{G_\beta}(L_2)$. So

$$\begin{aligned} s_i &= |P_{V_i}(L_1 \otimes L_2)/(V_i \cap (L_1 \otimes L_2)| \\ &= (s'_0)^{\beta+1} \cdot (s''_\beta). \end{aligned}$$

Now, we prove (3). For any $0 \le i < mn$, $i = \alpha m + \beta$, $0 \le \beta < m$,

$V_i = (span(w_0, \cdots, w_{\alpha-1}) \otimes span(u_0, \cdots, u_{m-1})) \oplus (span(w_\alpha) \otimes span(u_0, \cdots, u_\beta)) = (span(a_0, \cdots, a_{\alpha-1}) \otimes span(b_0, \cdots, b_{m-1})) + (span(a_\alpha) \otimes span(b_0, \cdots, b_\beta))$.

Then $V_i \cap (L_1 \otimes L_2) = L(a_0 \otimes b_0, \cdots, a_0 \otimes b_{m-1}, \cdots, a_{\alpha-1} \otimes b_{m-1}, a_\alpha \otimes b_0, \cdots, a_\alpha \otimes b_\beta)$.

On the other hand, $P_{V_i}(L_1 \otimes L_2) \subseteq (P_{span(w_0,\cdots,w_{\alpha-1})}(L_1) \otimes L_2) \oplus (P_{span(w_\alpha)}(L_1) \otimes P_{span(u_0,\cdots,u_\beta)}(L_2))$.

Let $\overline{L} = (P_{span(w_0,\cdots,w_{\alpha-1})}(L_1) \otimes L_2) \oplus (P_{span(w_\alpha)}(L_1) \otimes P_{span(u_0,\cdots,u_\beta)}(L_2))$. Clearly, $|P_{V_i}(L_1 \otimes L_2)/(V_i \cap (L_1 \otimes L_2))| \le |\overline{L}/(V_i \cap (L_1 \otimes L_2))|$. We determine $|\overline{L}/(V_i \cap (L_1 \otimes L_2))|$ as follows:

Let $L' = L(a_0 \otimes b_0, \cdots, a_0 \otimes b_{m-1}, \cdots a_{\alpha-1} \otimes b_{m-1}, w_\alpha \otimes b_0, \cdots, w_\alpha \otimes b_\beta)$.

Then $L' = (L(a_0, \cdots, a_{\alpha-1}) \otimes L_2) \oplus (L(w_\alpha) \otimes L(b_0, \cdots, b_\beta))$. Since $(w_0, w_1, \cdots, w_{n-1}) = (a_0, a_1, \cdots, a_{n-1})P$,$|(V_i \cap (L_1 \otimes L_2))/L'| = (p_{\alpha\alpha})^{\beta+1}$.

Since $|(P_{span(w_0,\cdots,w_{\alpha-1})}(L_1) \otimes L_2)/(L(a_0, \cdots, a_{\alpha-1}) \otimes L_2)| = (s'_{\alpha-1})^m$ and $|(P_{span(w_\alpha)}(L_1) \otimes P_{span(u_0,\cdots,u_\beta)}(L_2))/(L(w_\alpha) \otimes L(b_0, \cdots, b_\beta))| = (g'_\alpha)^{\beta+1} s''_\beta$,

$$|\overline{L}/L'| = (s'_{\alpha-1})^m \cdot (g'_\alpha)^{\beta+1} s''_\beta.$$

So, $|\overline{L}/(V_i \cap (L_1 \otimes L_2))| = (s'_{\alpha-1})^m \cdot (\frac{g'_\alpha}{p_{\alpha\alpha}})^{\beta+1} \cdot (s''_\beta)$. Hence,$s_i = |P_{V_i}(L_1 \otimes L_2)/(V_i \cap (L_1 \otimes L_2))| \le (s'_{\alpha-1})^m \cdot (\frac{g'_\alpha}{p_{\alpha\alpha}})^{\beta+1} \cdot (s''_\beta)$. By the Lemma 2.1, $d^-_\alpha = p_{\alpha\alpha}$ and so the proof is finished. ■

Remark 2.9 *V. Tarokh gave a very good upper bound on label complexity function (Theorem 5.1) by Lemma 5.3 of [7]. Theorem 2.8 gives the relations among the numbers of the states at corresponding levels in trellis diagrams of L_1, L_2 and $L_1 \otimes L_2$. Could we give a better upper bound on state and branch complexity functions by Theorem 2.8?*

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