Design of feedback dissipativity for transportation system

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Abstract : - This paper studies the traffic flow controlled by traffic lights on a single intersection. Firstly, a discrete-time model that describes the evolution of queue length will be given. Secondly, we show that, by regarding the flow entry of vehicles as the energy stored in the system, the dissipativity concept can be applied. As a result, we derive a feedback control law which renders our system dissipative, i.e. it dissipates the flow of vehicles in the intersection. Finally, an example is worked out.

Key-Words : Traffic control, Discrete-time linear system, Dissipative system, Feedback dissipativity.

1 Introduction

As the number of vehicles and the need for transportation grow, cities around the world face serious road traffic congestion problems. Costs include lost work and leisure time, increased fuel consumption, air pollution, health problems.... In general, there exist different methods to tackle the traffic congestion problem. The most effective measures in the battle against traffic congestion seem to be a selective construction of new roads and a better control of traffic through traffic management. Traffic light control can be used to augment the flow of traffic in urban environments by providing a smooth circulation of the traffic or to regulate the access to highways or main roads. The purpose of traffic lights is to provide efficient interaction of vehicles within the intersection. The goals of safety and efficiency are met when the delay for each vehicle and the number of accidents are kept to a minimum. We are interested in investigating the influence of better traffic light control policies on controlling or preventing traffic jams in urban environments. Currently, most of the control methodology is based on rigidly transferring the access right from lane to lane in a preprogrammed periodic cyclic rhythm. This approach is easy to implement but may be not efficient and flexible because it does not take into account traffic changes.

2 System description and modelling

An intersection is defined as a node and a segment that connects two nodes as a link (a vector imposed with a direction). A two-way street is represented by two links. Each node is operated by a traffic signal that can be assigned a 2-phase, 3-phase or 4-phase [9] operating system. At an intersection, the traffic signal cycle times are divided into different phases. Each phase is allotted a certain amount of time (green time) during which a group of traffic lanes is allowed to proceed. The movement may include vehicles going straight (through), turning left, turning right, or a combination of them.

2.1 Systems description

We consider an isolated two-phase intersection with controllable traffic lights on each corner (Figure 1). It is supposed that there are two traffic flows of vehicles to be served in the intersection. Hence, movements 1 and 2 in each phase have the same characteristics. Therefore, we are interested only on one movement of flow in each phase. The same transformation will be considered for movements 3 and 4. Traffic signal control separates traffic streams in an intersection by allocating different time intervals for conflicting traffic movements. For each movement the signals are given cyclically in the following order: Red, Green and Yellow. A green interval is followed by a yellow change interval indicating that a vehicle must stop if it can done safety.

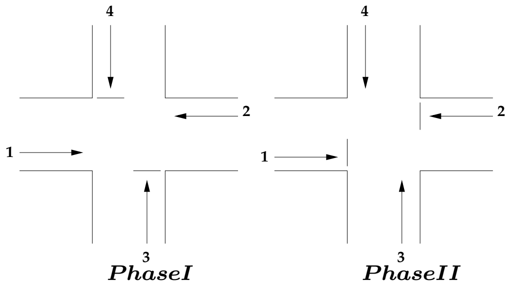

Fig. 1: Four-leg intersection with two-phase signal control.

To be able to present the model, a certain number of definitions are necessary:

Effective green time: as a vehicle approaches an intersection displaying a red signal, the driver decelerates, and stops either at the stop line or at the end of a queue (Figure 2). When the signal turns green, the driver accelerates until the vehicle reaches its desired or maximum possible speed. It is usually [2], [10] assumed that after startup lost time the saturation flow rate remains constant until the beginning of the yellow change interval. The effective green time is defined by:

$$g_e = g + y - l = g + y - (l_1 + l_2) \tag{1}$$

where the lost time l is the sum of startup lost time l_1 and clearance lost time l_2, y represents the interval of yellow light.

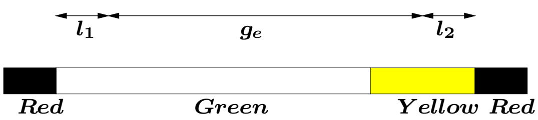

Fig. 2: Effective green time.

Saturation flow rate: the discharge process of the vehicles in the queue is controlled by the reaction times and desired acceleration rates of drivers as well as the acceleration rates of vehicles ahead. At the beginning of the green interval the discharge rate at stop lane starts to increase. As the queuing vehicles have reached a constant speed at stop line the discharge rate has reached its maximum, called the saturation flow rate. More precisely, the saturation flow rate is defined as the maximum number of vehicles being able to use the corridor without interruption during the effective time of the green light g_e. The saturation flow rate may vary from cycle to cycle, but an average value can be used for given conditions.

Oversaturated conditions: the oversaturated conditions is defined when arrival rate exceeds capacity in the intersection. A phase is saturated when a vehicle at least is constrained to await more than one cycle to cross the crossroads. The crossroads is saturated when at least one of its phases is saturated. Formally, the number of arrivals $E(k)$ during a cycle $k \in \mathbb{N}$ is defined by the following equation:

$$E(k) = E(kc) - E\big((k-1)c\big) \tag{2}$$

Where c defined as length cycle and $E(kc)$ defined as the number of the arrivals at the end of the cycle. The departure rate or saturation flow rate s during the effective time of the green light g_e is defined by :

$$D(k) = s\, g_e \tag{3}$$

where $D(k)$ is the number of departures at the end of the effective green time g_e. Figure 3 displays a case where demand flow rate instantaneously increases above the capacity at the beginning of a cycle. The capacity curve $C(t)$ is not the saw-toothed departure curve $D(t)$, so that the area between $E(t)$ and $C(t)$ curves is the overflow delay [4]. The following inequality summarizes the definition of the oversaturation condition:

$$E(k) > D(k) \tag{4}$$

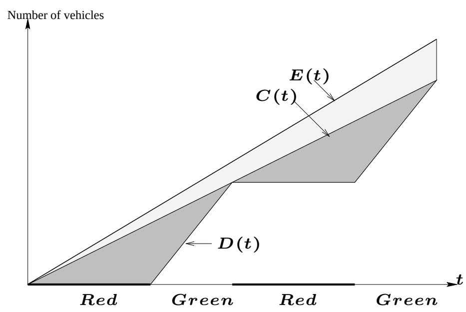

Fig. 3: Model for overflow delay.

2.2 Modelling

The conventional delay formula can be modified into a state-dependent form called state space equations. An optimal control methodology can then be applied to determine optimal timing from the constructed state space of an intersection. The proposed model is formulated as a discrete type operation. [1], [3], [4], [5], [6] and [7] constructed similar models for oversaturation control. But their models are all continuous ones. However, continuous models are limited in the sense that the switch-over point does not necessarily occur at the end of a cycle, and the termination of the oversaturated period occur only at the end of the final cycle. On the other hand, the switch-over points determined by a discrete model occur exactly at the termination of a cycle. Discrete operation provides a smooth, regular, and ordered transfer of control. Calculating queue is more reliable. In the case of a cross intersection with a two-phase signal (see Figure 1), during the oversaturated period the queue and dispersion situation is as indicated in Figure 3. Without loss of generality, it is assumed herein that the cumulative demand on all approaches is a linear asymptotic function of time and that the cumulative output curves do not intersect the cumulative input curves for any of the approaches. This fact implies that no queue becomes negative or zero before the end of the oversaturated period. Figure 4 shows both situations of queue during oversaturation and the duality between the two phases. To keep this duality, it is necessary that the queue represented during the effective green time terminates at a certain cycle state $\boldsymbol{k}$.

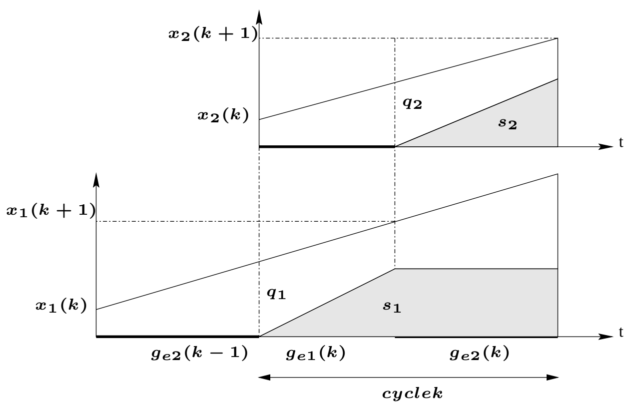

Fig. 4: Queue of a four-leg intersection with two-phase control.

let $\boldsymbol{x_1(k)}$ be a queue length of approach 1 when the effective green time $\boldsymbol{g_{e1}(k) = c - g_{e2}(k)}$ terminates at a certain cycle state $\boldsymbol{k}$, according to Figure 4 the relation of the queue lengths between cycle $\boldsymbol{k}$ and $\boldsymbol{k+1}$ can be represented by the following equation:

$$\boldsymbol{x_1(k+1) = x_1(k) + E_1(k) - D_1(k)} \qquad (5)$$

where $\boldsymbol{E_1(k) = q_1\, g_{e2}(k-1) + q_1\,(c - g_{e2}(k))}$ represents the number of arrivals at the end of $\boldsymbol{c - g_{e2}(k)}$. $\boldsymbol{D_1(k) = s_1\,(c - g_{e2}(k))}$ represents the number of departures at the end of $\boldsymbol{c - g_{e2}(k)}$. Hence, equation $\boldsymbol{(5)}$ becomes:

$$\begin{aligned}\boldsymbol{x_1(k+1)} &= \boldsymbol{x_1(k) + q_1\, g_{e2}(k-1)} \qquad (6)\\ &\quad \boldsymbol{+\, q_1\,(c - g_{e2}(k)) - s_1\,(c - g_{e2}(k))}\end{aligned}$$

Similarly, let $\boldsymbol{x_2(k)}$ be a queue length of approach 2 when the effective green time $\boldsymbol{g_{e2}(k)}$ terminates at a certain cycle state $\boldsymbol{k}$. Then the relation of the queue lengths between states $\boldsymbol{k}$ and $\boldsymbol{k+1}$ can be represented by the following equations:

$$\boldsymbol{x_2(k+1) = x_2(k) + E_2(k) - D_2(k)} \qquad (7)$$

where $\boldsymbol{E_2(k) = q_2 c}$ and $\boldsymbol{D_2(k) = s_2\, g_{e2}(k)}$ represent the number of arrivals and the number of departures during cycle $\boldsymbol{k}$ respectively. Hence, equation (7) takes the form:

$$\boldsymbol{x_2(k+1) = x_2(k) + q_2\, c - s_2\, g_{e2}(k)} \qquad (8)$$

Let $\boldsymbol{x(k)} \in \mathbb{R}^{\boldsymbol{2}}$ be the state vector defined as $\boldsymbol{x(k) = \big(x_1(k), x_2(k)\big)^t}$, and $\boldsymbol{\epsilon(k)} \in \mathbb{R}^{\boldsymbol{2}}$ be the control vector defined by $\boldsymbol{\epsilon(k) = \big(g_{e2}(k), g_{e2}(k-1)\big)^t}$. Then we can gather the two equations $\boldsymbol{(6)}$ and $\boldsymbol{(8)}$ in the following matrix form:

$$\boldsymbol{x(k+1) = A\, x(k) + B\, \epsilon(k) + C} \qquad (9)$$

where

$$\boldsymbol{x(k)} = \begin{pmatrix} \boldsymbol{x_1(k)} \\ \boldsymbol{x_2(k)} \end{pmatrix} \ \boldsymbol{\epsilon(k)} = \begin{pmatrix} \boldsymbol{g_{e2}(k)} \\ \boldsymbol{g_{e2}(k-1)} \end{pmatrix}$$

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{1} \end{pmatrix} \ \boldsymbol{B} = \begin{pmatrix} \boldsymbol{s_1 - q_1} & \boldsymbol{q_1} \\ \boldsymbol{-s_2} & \boldsymbol{0} \end{pmatrix}$$

$$\boldsymbol{C} = \begin{pmatrix} \boldsymbol{(q_1 - s_1)\, c} \\ \boldsymbol{q_2\, c} \end{pmatrix}$$

Note that $\boldsymbol{B}$ is nonsingular for all $\boldsymbol{q_1, s_1, s_2}$, and $\boldsymbol{C}$ is constant vector. With this in mind, system (9) can be rewritten as

$$\boldsymbol{x(k+1) = Ax(k) + B\big(\epsilon(k) + B^{-1}C\big)} \qquad (10)$$

with the preliminarily control $\boldsymbol{u(k) = \epsilon(k) + B^{-1}C}$, then system (9) takes the simple form

$$\boldsymbol{x(k+1) = Ax(k) + Bu(k)} \qquad (11)$$

Notice that it makes no sense to speak of negative queue. Hence, if $\boldsymbol{x_0}$ describes queue at time $\boldsymbol{k = 0}$ then $\boldsymbol{x(k) \geq 0}$ for all $\boldsymbol{k} \in \mathbb{N}$. Therefore, the system (11) has physical meaning only if $\boldsymbol{x}$ belongs to the region of admissible states $\boldsymbol{\Omega_x = \{x} \in \mathbb{R}^{\boldsymbol{n}} \boldsymbol{/ x \geq 0\}}$.

3 Dissipativity approach

This section focus on the study of the energy of systems in terms of their dissipativity property. Dissipativity and its particular case of passivity were born from the observation of physical systems behavior. The energy concept is very useful in the analysis of physical systems. Many systems can be studied from its sources and of losses of energy. Having the idea of the gain and the loss of energy, intuitively, a dissipative system is such a system which cannot store all energy that has been given. A dissipative systems dissipate energy and does not produce it, that is, any increase of stored energy is only due to external sources. The definition of dissipative system based on the existence of three energy like functions: storage function (representing the energy stored by the system), a supply function (the energy injected to the system from an external source, which restricts the manner in which the system absorbs energy) and the dissipation function (representing the total energy dissipated by the system in some time interval). Depending upon the form of the supply function, different kinds of dissipativity are obtained; passivity is the one which attracted more attention. The one who defined dissipativity concepts by means of the notion of the storage, the supply rate and the dissipation rate function was *Willems* in the early 70's [11]and [12].

3.1 Feedback dissipativity through the dissipativity equality

We begin by introducing a number of basic definitions and concepts related to the notions of dissipativity and passivity. Let a linear discrete-time system of the form

$$x(k+1) = Ax(k) + Bu(k) \tag{12}$$

$$y(k) = h(x(k), u(k)), \tag{13}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$. A and B are appropriately dimensioned matrices. Assume that a function $s(\cdot)$ defined on $\mathbb{R}^p \times \mathbb{R}^m$ is given. This function is called the supply rate. The following definitions are taken from [8].

Definition 1 *A dynamic system* (12)-(13) *with supply rate* $s(\cdot)$ *is said to be dissipative if there exists a nonnegative function* $V : \mathbb{R}^n \longrightarrow \mathbb{R}$, *with* $V(0) = 0$, *called the storage function and a continuous function* $\phi : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}$, *with* $\phi(\cdot, u)$ *positive for each* $u \in \mathbb{R}^m$, *regarded as the dissipation rate function, such that for all* $u \in \mathbb{R}^m$ *and all* $k \in \mathbb{N}$.

$$V(x(k+1)) - V(x(k)) = s(y(k), u(k)) - \phi(x(k), u(k)) \tag{14}$$

Definition 2 *A system of the form* (12)-(13) *is said to be passive if it is dissipative whit* $s = y^t u$.

Definition 3 *Consider the system* (12)-(13) *and two scalar functions* $V(x)$ *and* $s(y, v)$ *as a storage function and a supply function, respectively. The system is said to be feedback dissipative with the functions* V *and* s , *if there exists a regular static state feedback control law of the form* $u = \alpha(x, v)$, *with* v *as the new input, such that the feedback transformed system is dissipative with respect to* v.

Definition 4 *A system of the form* (12)-(13) *is said to be feedback passive if it is feedback dissipative whit* $s = yv$.

The existence of the feedback control law of the form $u = \alpha(x, v)$ for with the system is rendered dissipative must be assessed from the existence of solutions, for the control input u, of the following equation:

$$V(x(k+1)) - V(x(k)) = s(h(x(k), u(k)), v(k)) - \phi(x(k), u(k)) \tag{15}$$

The work of (Navarro-Lopez et al) ensure the sufficient conditions under which feedback dissipativity is possible [8].

3.2 Application of passifying methodology to queues evolution model

The feedback dissipativity scheme defined in preceding section will be applied to the passivation of our system (11). Indeed, the intersection is regarded as being a system which receives the external energy. This energy is exactly the flow of the vehicles which feeds our system. The increase of the flow generates an increase on queues, therefore a dissipation of energy (external flow) implies a minimization on the level of the queues. The energy associated to the system will be used as storage function. The feedback dissipativity methodology is then applied with $s = yv$ and $V = x^t x$. The output of the system is considered to be $y = x$. First of all, a function ϕ must be proposed. This function will be chosen in order to collect the positive terms in $V(x(k+1))$. The control which passifies the system will be then obtained from the equation:

$$V(x(k+1)) - V(x(k)) = y(k)v(k) - \phi(x(k), u(k)) \tag{16}$$

with v the new control input. Hence, for our system, equation (15) becomes,

$$\phi(x, u) = x^t v + x^t x - (x + Bu)^t (x + Bu)$$

which implies

$$\phi(x, u) = x^t v - 2x^t Bu - u^t B^t Bu \tag{17}$$

Consequently, a possibility for ϕ is the following one $\phi(x, u) = u^t B^t Bu + x^t Qx$, where Q is positive definite matrix. Then, from the equation **(17)** the control which passifies the system will be derived as follow:

$$u(k) = B^{-1}(\frac{1}{2}v(k) - Qx(k)) \qquad (18)$$

It is important to note that for $v = 0$, equation (15) yields

$$V(x(k+1)) - V(x(k)) = -\phi(x(k), \alpha(x, 0)) \qquad (19)$$

Since V and ϕ are positive definite, it follows that the feedback control $\alpha(x, 0)$ stabilizes the system. Hence, to achieve this fact with our proposed feedback dissipativity, the matrix Q must be chosen such that the matrix $I - Q$ be a stable one. To do this observe that for all matrix M of order n with eigenvalues λ_i, it spectral radius $\sigma(M) = \max_i |\lambda_i|$ verifies

$$\sigma(M) = \max_i \sum_{j=1}^{n} |m_{ij}|$$

Hence, it suffice to choose Q such that

$$\max_i \sum_{j=1}^{n} |\delta_{ij} - Q_{ij}| < 1$$

where δ_{ij} is the kronecker symbol. Summarizing the above results give the following Proposition.

Proposition 1 *Consider the transportation system* **(11)**. *Let* $V = x^t x$ *and* $s = yv$ *be the storage function and a supply function respectively. Then the feedback control* $u(k) = B^{-1}(\frac{1}{2}v(k) - Qx(k))$*, with* Q *being positive definite such that*

$$\max_i \sum_{j=1}^{n} |\delta_{ij} - Q_{ij}| < 1$$

renders the feedback transformed system passive with v *as the new input.*

The following simulations illustrate the application of the feedback passitivity to our system. The first simulation run represents the dynamic of the queue length when $v(k) = 0$. We can immediately observe that the feedback control law $u = -B^{-1}Qx$ stabilizes the system.

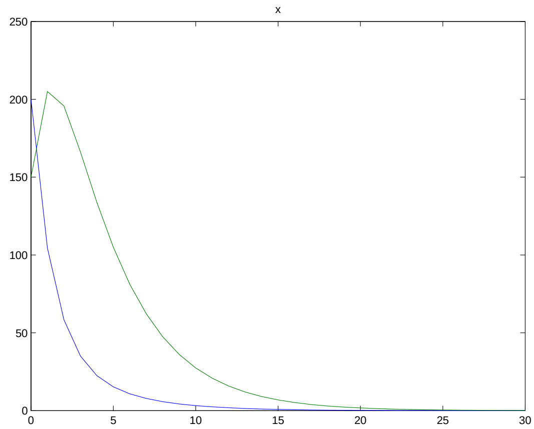

Fig. 5: Stability of the queue evolution system

The second simulation run represents the evolution of the energy (the norm of the queue length) $V = x^t x$ stored into the system. We can observe that the application of feedback passitivity renders our system dissipative. In other word, the control makes the queue length less than a certain level. Therefore, the system can't be oversaturated.

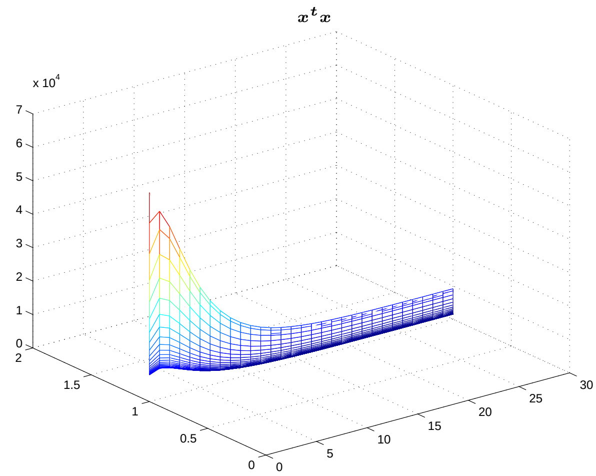

Fig. 6: Energy dissipation of queue evolution system

We notice that the application of the dissipation strategy through the feedback control dissipation makes the system stable and it dissipates the energy stored into the system. The only requirement to apply this strategy is the choice of matrix Q such that the condition given in Proposition 1 is verified.

4 Conclusion

In this paper, we have proposed a modelling which manages queues length in an insolated intersection. We have introduced the concept of dissipativity by considering the flow entry of vehicles as being the energy stored in the system. Through the dissipativity equality we found a feedback control law that renderers the system dissipative, i.e., the system can't be oversaturated.

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